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# A 14 Factor Heath, Jarrow and Morton Model for the German Bund Yield Curve, January 1996 to March 2017

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## ABSTRACT

This paper analyzes the number and the nature of factors driving the movements in the German Bund yield curve from January 1, 1996 through March 27, 2017. The process of model implementation reveals a number of important insights for interest rate modeling generally. First, model validation of historical yields is important because those yields are the product of a third-party curve fitting process that may produce spurious indications of interest rate volatility. Second, quantitative measures of smoothness and international comparisons of smoothness provide a basis for measuring the quality of historical and simulated yield curves. We find that the yield curve smoothing by the third-party vendor used for Germany required careful vetting. Third, we outline a process for incorporating insights from the Japanese experience with negative interest rates into term structure models with stochastic volatility in Germany and other countries. Finally, we illustrate the process for comparing stochastic volatility and affine models of the term structure. We conclude that stochastic volatility models have a superior fit to the history of yield movements in the German Bund market.

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**A 14 Factor Heath, Jarrow and Morton Model for the German Bund Yield Curve, January 1996 to March 2017**

Government yield curves are a critical input to the risk management calculations of central banks, bank regulators, major banks, insurance firms, fund managers, pension funds, and endowments around the world. With the internationalization of fixed income investing, it is important to understand the dynamics of movements in yield curves world-wide, in addition to the major bond markets like those in Frankfurt, London, New York and Tokyo. In this paper, we fit a multi-factor Heath, Jarrow and Morton model to daily data from the German Bund market over the period from January 1, 1996 to March 27, 2017. The modeling process reveals a number of important implications for term structure modeling in other government bond markets.

Section I discusses the origin and characteristics of the daily data base of German Bund yields provided by Bloomberg. We present a video of daily yield curve movements after overlaying the maximum smoothness forward rates (see Adams and van Deventer, 1994, as modified in van Deventer and Imai, 1996). We also compare the smoothness measures of German Bund yield curves with the smoothness of government yield curves in 9 other countries. We conclude that Bloomberg time series is realistically smooth and a reliable foundation for term structure modeling after eliminating one erroneous observation. This contrasts with recent findings from Japan and Thailand where we found that yield curves were unrealistically rough and that modification of the input data was necessary for a realistic model, a standard aspect of a Bayesian model validation process.

Section II outlines the process for determining whether the interest rate volatility for the factors driving the German Bund yield curve is constant (an "affine" model) or stochastic, typically expressed as a function of the level of interest rates. We note the extensive experience with negative interest rates in the Japanese government bond market and use insights from that experience in fitting volatility in the German Bund market, where negative rates have also been common. Section III describes the process of fitting five different Heath, Jarrow and Morton models to German Bund yield data: models with 1, 2, 3, 6 and 14 factors. We conclude Section III with extensive Bayesian model validation procedures, including stress tests of the derived interest rate volatility functions and the empirical drift in forward rates. We close the section by giving an example, using U.S. Treasury data, of a 30-year forward-looking simulation of 250,000 scenarios. Section IV concludes the paper. The Appendix illustrates a sample model validation process for widely used one factor term structure models using German data.

## I. German Bund Data: Special Characteristics

A multi-factor term structure model is the foundation for best practice asset and liability management, market risk, economic capital, interest rate risk in the banking book, stress-testing and the internal capital adequacy assessment process. The objective in this paper is to illustrate the derivation of a multi-factor Heath Jarrow and Morton model of the German Bund yield curve. As a by-product, the analysis has the potential to detect common data problems associated with yield curve histories and employs a standard methodology for quantification and resolution of those problems. Previous implementations of multi-factor Heath, Jarrow and Morton models have covered the following bond market sectors:

| | |
|---|---|
| Australia | Commonwealth Government Securities |
| Canada | Government of Canada Securities |
| Germany | German Bunds* |
| Japan | Japanese Government Bonds |
| Singapore | Singapore Government Securities |
| Spain | Spanish Government Bonds |
| Sweden | Swedish Government Securities |
| Thailand | Thai Government Securities |
| United Kingdom | United Kingdom Government Bonds |
| United States | U.S. Treasury Securities |

* Prior version.

The first step in data model validation for the German Bund market is to examine the historical availability of bond yields over time. For ease of use reasons, we used data provided by Bloomberg rather than data provided by the Deutsche Bundesbank. For the entire time period, Bloomberg data was available out to 30 years, so only one data regime needed to be considered.

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Because the Heath, Jarrow and Morton analysis makes use of a yield curve with quarterly forward rate segments, the next step in data model validation is to fit quarterly forward rates to the raw coupon-bearing bond yields. The smoothness of the resulting forward rates will be a function of both the quality of the raw data from a smoothness point of view and the smoothness implied by the secondary smoothing process. To ensure the maximum smoothness from the secondary smoothing process, we use the maximum smoothness forward rate methodology of Adams and van Deventer [1994], as corrected in van Deventer and Imai [1996]. Adams and van Deventer show that the maximum smoothness method overcomes the problems of the cubic spline approach of McCulloch, and, unlike the Svensson [1994] approach, allows for a perfect fit to the coupon-bearing bond yield data provided by Bloomberg. See Jarrow [2014] for information on the problems with Svensson yield curve fitting.

We then conduct a visual inspection of the resulting forward rates implied by the raw data. A video of the daily quarterly forward rates (in red) versus the zero-coupon bond yields (blue) implied by the German Bund data on every business day from 1996 through 2017 is given here:

https://www.youtube.com/watch?v=aPrP6E-V2KQ&t=39s

The smoothness of the quarterly forward rate curve can be measured quantitatively using the quarterly forward rates implied by the German Bund yield curves. For a yield curve that consists of N quarterly forward rates, the discrete smoothness statistic at time t $Z_N(t)$ is the sum of the squared second differences in the forward rates, as explained by Adams and van Deventer [1994]. A closed form continuous smoothing statistic can also be calculated when the functional form of the continuous forward rate is known.

$$Z_N(t) = \sum_{i=3}^{N} \left[ \left( f_i(t) - f_{i-1}(t) \right) - \left( f_{i-1}(t) - f_{i-2}(t) \right) \right]^2$$

A statistical comparison of smoothness for German Bund yields with yields from 9 other government curves reveals the following:

**Table I**

| Discrete Smoothing Index Values for Government Securities Yield Data | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 10 Year Yield Curve | | | | 30 Year Yield Curve | | | |
| Yield Curve | Observations | Mean | Median | Maximum | Observations | Mean | Median | Maximum |
| United Kingdom Government Bond | 9,627 | 0.0729 | 0.0260 | 75.4613 | 4,959 | 0.0418 | 0.0166 | 1.6158 |
| Japanese Government Bond | 7,434 | 0.0023 | 0.0010 | 0.0752 | 4,241 | 0.0010 | 0.0008 | 0.0755 |
| U.S. Treasury | 13,862 | 0.1767 | 0.0372 | 17.2179 | 9,098 | 0.2470 | 0.0823 | 17.2213 |
| German Bund | 6,612 | 0.3510 | 0.0735 | 1146.8713 | 6,612 | 0.3691 | 0.0764 | 1239.0629 |
| Australian Government Securities | 6,612 | 0.7022 | 0.0339 | 1463.5491 | 6,612 | 0.7500 | 0.0359 | 1603.3647 |
| Canada Government Securities | 6,145 | 0.2895 | 0.0742 | 11.6386 | 6,145 | 0.2910 | 0.0749 | 11.6455 |
| Singapore Government Securities | 4,654 | 0.0322 | 0.0155 | 0.6499 | 1,244 | 0.0130 | 0.0097 | 0.1767 |
| Swedish Government Securities | 7,554 | 0.3366 | 0.0914 | 152.2298 | 7,554 | 0.3366 | 0.0914 | 152.2298 |
| Thai Government Bond | 4,236 | 0.0979 | 0.0417 | 5.8143 | 1,435 | 0.0470 | 0.0256 | 0.3595 |
| Russia Government Securities | 3,559 | 0.1609 | 0.0464 | 3.4501 | 3,559 | 0.1609 | 0.0464 | 3.4501 |
| Average | | 0.2222 | 0.0441 | | | 0.2257 | 0.0460 | |

The maximum smoothing index values for 10 and 30 years for the German Bund market are the largest reported world-wide except for Australia. When we rank the observation dates by highest smoothing index values, we find that one bad observation has been caused by a data vendor error.

| | |
|---|---|
| June 15, 1998 | 1146.8713 |
| October 15, 2008 | 4.2465 |
| October 7, 2008 | 3.1828 |
| March 15, 1996 | 3.1069 |
| March 16, 1996 | 3.1069 |

This video makes the daily comparison from 1996 to 2016 of forward rates for German Bunds and U.S. Treasuries. The impact of the bad data point is quite striking:

https://www.youtube.com/watch?v=NeWD2I8T-KI&list=PLFtDZOVCnk_oKk8bC4OnwQ_U94MCu9_EL&index=8

Accordingly, we drop the bad observation and review the distribution of smoothing statistics for the remaining German Bund yield curve data:

**Exhibit I: 10-year Smoothing Statistic**

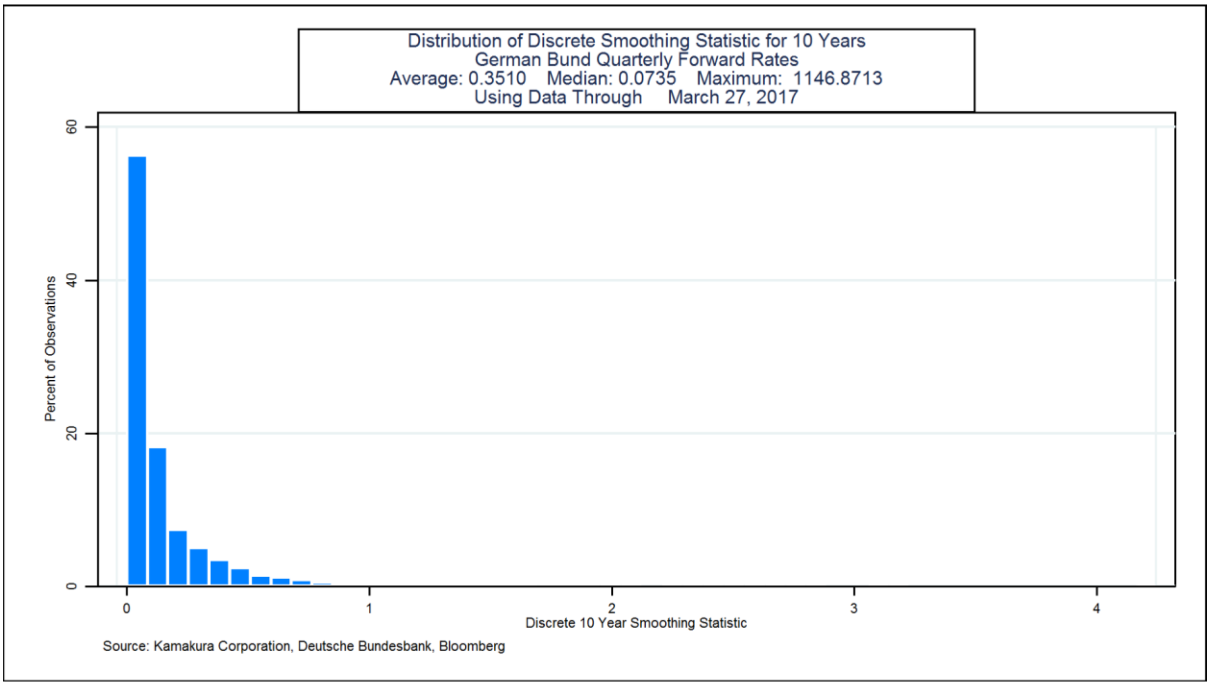
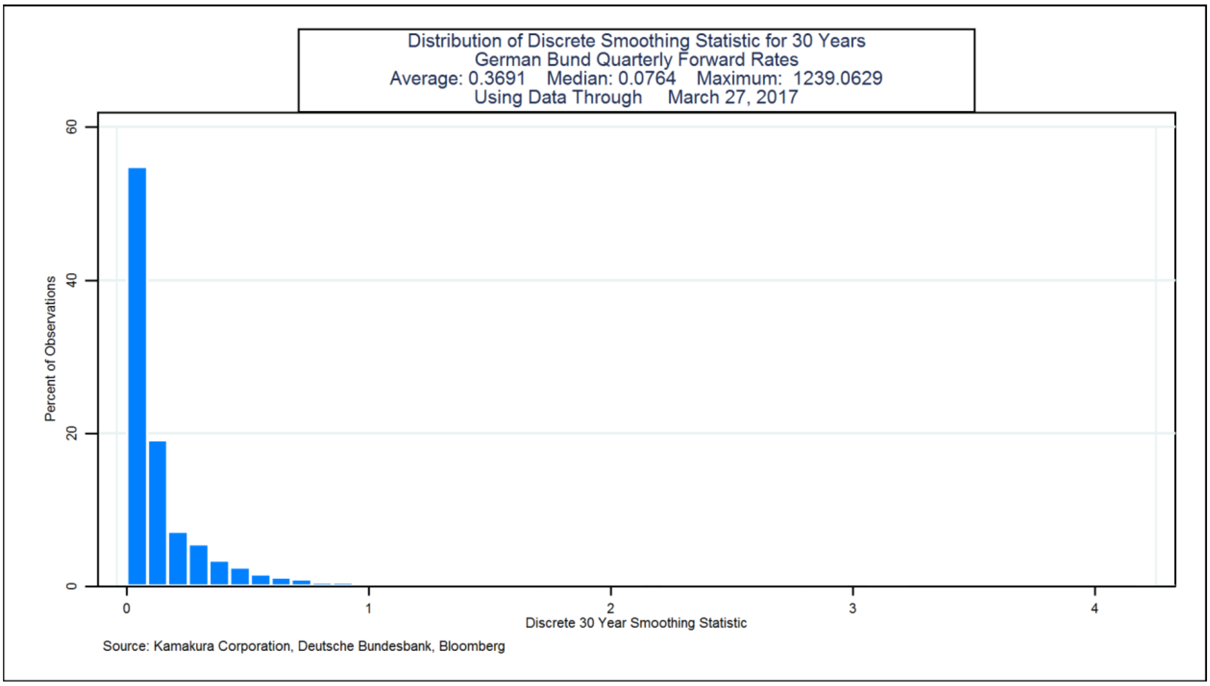



**Exhibit II: 30-year Smoothing Statistic**

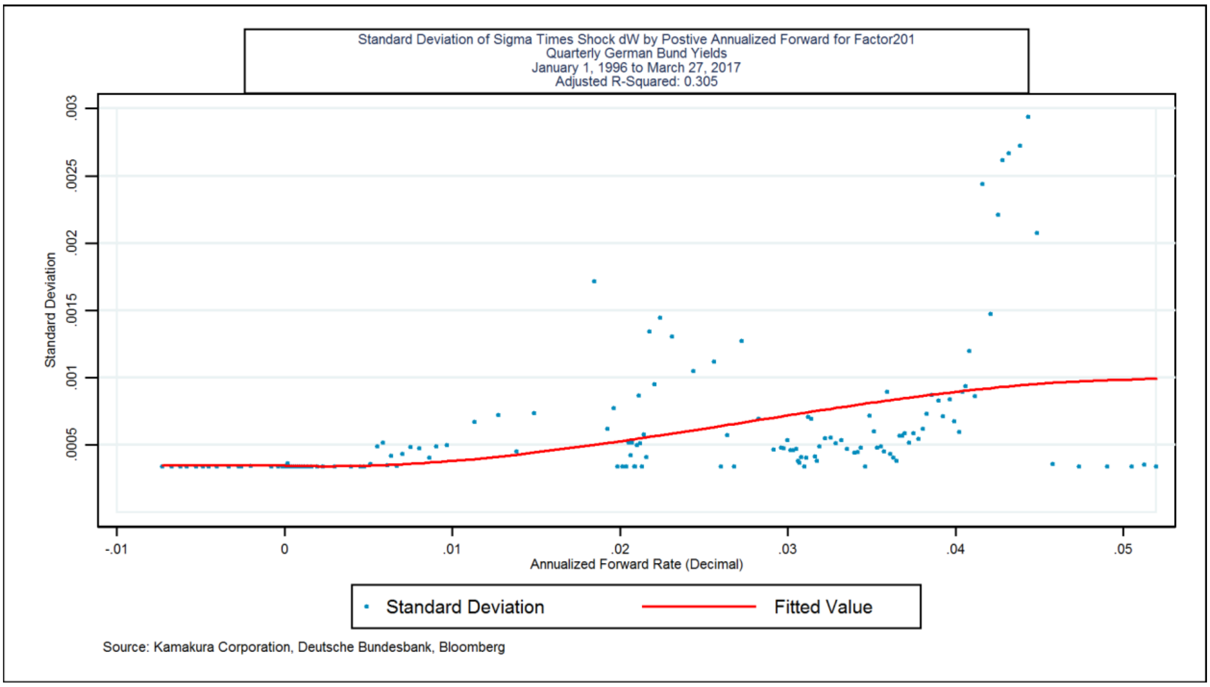



After removing the troubled observation, we conclude that the data provided by Bloomberg is realistic in both its smoothness and its movements, providing a sound basis for term structure modeling.

## II. Constant versus Stochastic Volatility

Constant volatility ("affine") term structure models are commonly used for their ease of simulation and estimation of "future expected rates" in order to determine the "term premium" in current yields. Prominent examples are Adrian, Crump and Moench [2013], Kim and Wright [2005], and Duffie and Kan [1996]. On the other hand, the weight of the empirical evidence in most of the countries studied to date indicates that interest rate volatility does vary by the level of the corresponding forward rate. To illustrate that fact, we studied the shortest forward rate on the German Bund curve on a daily basis from January 1, 1996 through March 27, 2017. We ordered the data from lowest forward rate level to highest forward rate level. We formed non-overlapping groups of 50 observations each and calculated both the standard deviation of 91-day forward return changes and the mean beginning-of-period forward rate in each group. The results are plotted in Exhibit III:

**Exhibit III**

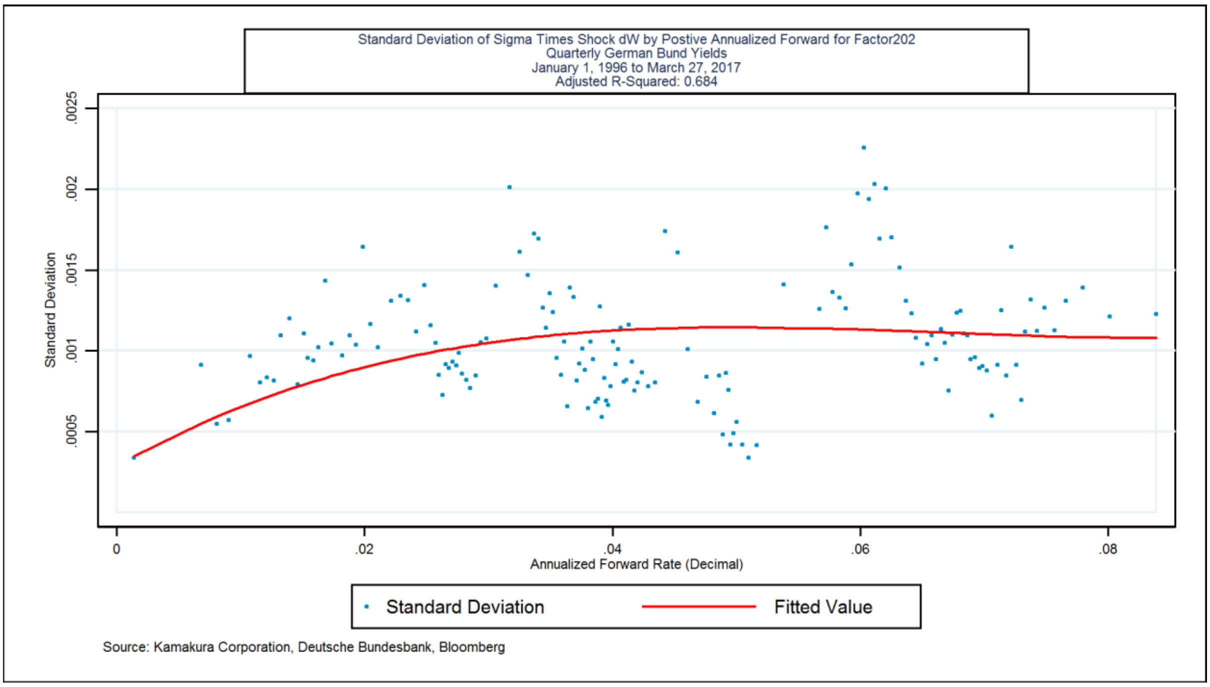



A cubic function of annualized forward rates explains 30.5% of the variation in the standard deviation of forward rate changes for these ordered groups. This is well below the average explanatory power for other government securities markets and raises questions again about the quality of the underlying data.[2] The red line is the volatility function used when extracting the first random factor driving the German Bund curve. Note that the right-hand side of the curve has been constrained to have a first derivative of zero at a high level of rates.[3] The rise in volatility in higher rate environments has been confirmed in the government securities markets for Australia, Canada, Japan, Singapore, Spain, Sweden, the United Kingdom, and the United States. Thailand, where interest rates have moved in a relatively narrow band, is the only exception so far. Exhibit IV shows the results for the second risk factor in the German Bund market, the idiosyncratic movements in the quarterly forward rate maturing in 30 years:

[2] We always prefer smoothing from raw coupon-bearing bond prices when such data is available. The Deutsche Bundesbank is one of the few central banks that makes such data available, but the time series was too short to employ in this study. We look forward to using the Deutsche Bundesbank data in the future.

[3] This constraint is one method for imposing the cap in stochastic volatilities suggested by Heath, Jarrow and Morton [Econometrica, 1992] to prevent a positive possibility of (a) infinitely high rates or (more practically) (b) unrealistically high rates.

**Exhibit IV**




The cubic stochastic volatility specification explains 68.4% of the observed idiosyncratic forward rate volatility in the quarterly forward rate maturing at the 30-year point on the German Bund yield curve. We have imposed the same constraint on the first derivative and require that the fitted volatility not be less than the observed volatility when interest rates are negative in Japan, which we discuss later in this section.

Exhibit V shows the historical movements in German Bund zero-coupon yields over the historical period studied:

**Exhibit V**

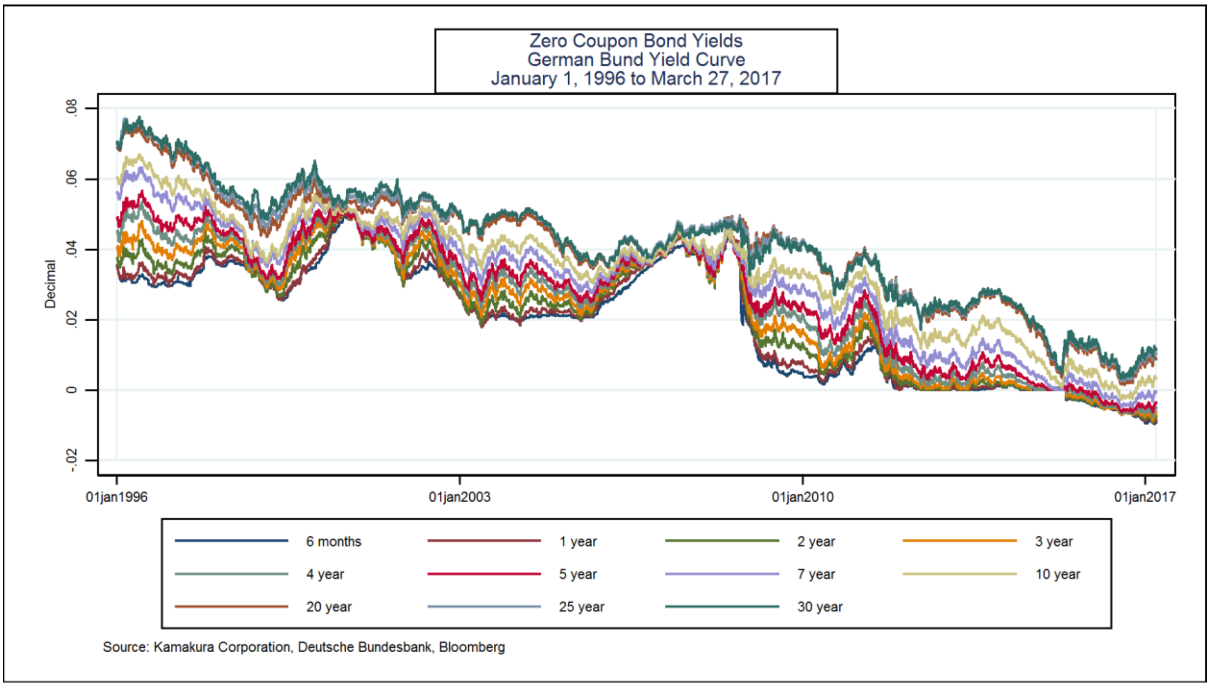
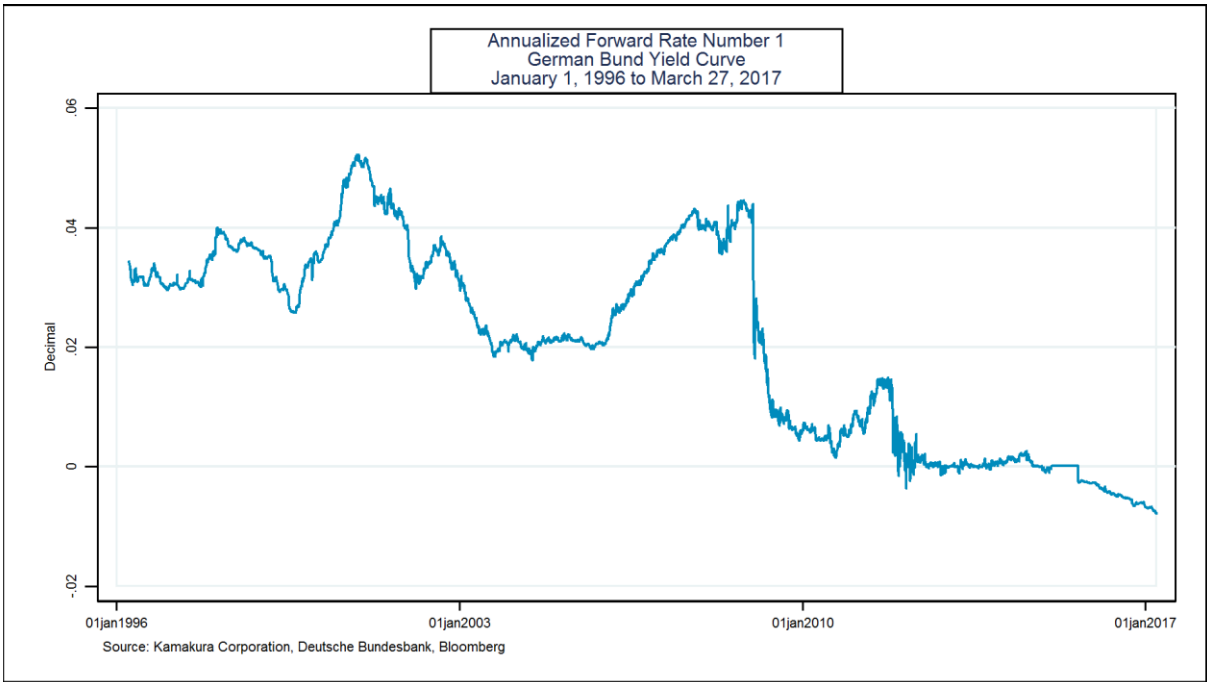


Exhibit VI below shows the evolution of the first quarterly forward rate (the forward that applies from the 91st day through the 182nd day) over the same time period:

**Exhibit VI**




We use three statistical tests to determine whether or not the hypothesis of normality for forward rates and zero-coupon bond yields should be rejected at the 5% level: the Shapiro-Wilk test, the Shapiro-Francia test, and the skew test, all of which are available in common statistical packages. The results of these tests are summarized in Table II:

Table II

**Kamakura Corporation**
**HJM 14 Factor Model**
**German Bund**
**Using Daily Data from January 1, 1996 through March 27, 2017**
**Date of Analysis: July 12, 2017**

**NORMALITY OF ZERO COUPON BOND YIELD ABSOLUTE LEVELS AND CONTINUOUSLY COMPOUNDED CHANGES IN FORWARD RETURNS**

| P-values for Null Hypothesis that Zero Coupon Bond Yields are Normally Distributed | | | | P-values for Null Hypothesis that Discrete Changes in Forward Returns Are Normally Distributed | | | | Z Yield Result | Return Result |
|---|---|---|---|---|---|---|---|---|---|
| Quarter | Shapiro Wilk Test | Shapiro Francia Test | Skew Test | Quarter | Shapiro Wilk Test | Shapiro Francia Test | Skew Test | Is Normality Hypothesis Rejected? | Is Normality Hypothesis Rejected? |
| 1 | 0.000% | 0.000% | | 1 | 0.000% | 0.000% | | Yes | Yes |
| 2 | 0.000% | 0.000% | | 2 | 0.000% | 0.000% | | Yes | Yes |
| 3 | 0.000% | 0.000% | | 3 | 0.000% | 0.000% | | Yes | Yes |
| 4 | 0.000% | 0.000% | | 4 | 0.000% | 0.000% | 0 | Yes | Yes |
| 5 | 0.000% | 0.000% | | 5 | 0.000% | 0.000% | 0 | Yes | Yes |
| 6 | 0.000% | 0.000% | | 6 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 7 | 0.000% | 0.000% | | 7 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 8 | 0.000% | 0.000% | | 8 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 9 | 0.000% | 0.000% | | 9 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 10 | 0.000% | 0.000% | | 10 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 11 | 0.000% | 0.000% | | 11 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 12 | 0.000% | 0.000% | | 12 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 13 | 0.000% | 0.000% | | 13 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 14 | 0.000% | 0.000% | | 14 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 15 | 0.000% | 0.000% | | 15 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 16 | 0.000% | 0.000% | | 16 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 17 | 0.000% | 0.000% | | 17 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 18 | 0.000% | 0.000% | | 18 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 19 | 0.000% | 0.000% | | 19 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 20 | 0.000% | 0.000% | | 20 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 21 | 0.000% | 0.000% | | 21 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 22 | 0.000% | 0.000% | | 22 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 23 | 0.000% | 0.000% | | 23 | 0.000% | 0.000% | 0.000% | Yes | Yes |
| 24 | 0.000% | 0.000% | | 24 | 0.000% | 0.000% | 0.000% | Yes | Yes |

**Number Rejected**

| Z Yield Result | Return Result |
|---|---|
| Is Normality Hypothesis Rejected? | Is Normality Hypothesis Rejected? |
| 120 | 118 |

Table II above shows the p-values for these three statistical tests for the first 24 quarterly maturities. We conduct the test for each quarter out to 30 years, the longest maturity used in the smoothing process. The null hypothesis of normality is rejected by all 3 tests for 120 of the 120 quarterly zero-coupon yield maturities. For quarterly changes in forward rates, the null hypothesis of normality is again rejected by all 3 tests for 118 of the 120 maturities for changes in forward rates. This is a powerful rejection of the normality assumptions implicit in constant coefficient or "affine" term structure models. In most of the other countries studied, the hypothesis of normality has been rejected strongly as well. Given these results, we proceed with caution on the implementation of the affine model.

In Chapter 3 of [Advanced Financial Management] (second edition, 2013), van Deventer, Imai and Mesler analyze the frequency with which U.S. Treasury forward rates move up together, down together or remain unchanged. This exercise informs the Heath, Jarrow and Morton parameter fitting process and is helpful for the model validation questions posed in the Appendix. We perform the yield curve shift analysis using 7,713 days of quarterly forward rates for the German Bund yield curve. We analyze the daily shifts in the forward rates on each business day from January 1, 1996 through March 27, 2017. The results are given in Table III:

Table III

| | | |
|---|---|---|
| **Kamakura Corporation** | | |
| **HJM 14 Factor Model** | | |
| **German Bund** | | |
| **Using Daily Data from January 1, 1996 through March 27, 2017** | | |
| **Date of Analysis: July 12, 2017** | | |
| **Type of Yield Shift** | **Number of Observations** | **Percent of Observations** |
| **All yields shift up** | 333 | 4.32 |
| **All yields shift down** | 363 | 4.71 |
| **All yields are unchanged** | 2,222 | 28.81 |
| **Yield curve twists** | 4,795 | 62.17 |
| **Total** | 7,713 | 100.00 |
| Kamakura Corporation, Deutsche Bundesbank, Bloomberg | | |

Yield curve shifts were all positive, all negative, or all zero 4.32%, 4.71%, and 28.81% of the time, a total of 37.84% of all business days. Note that the "all yields are unchanged" total is inflated because the Bloomberg data includes weekends. The predominant yield curve shift was a twist, with a mix of positive changes, negative changes, or zero changes. These figures are similar to those for the Japanese Government Bond, Government of Canada, United Kingdom Gilt and U.S. Treasury yield curves. These twists, which happen 62.17% of the time in the German Bund market, cannot be modeled accurately with the conventional implementation of one factor term structure models.

Another important aspect of yield curves is the number of local minima and maxima that have occurred over the modeling period. The results for the German Bund market are given in Table IV:

Table IV

**Analysis of Number of Local Minima and Maxima Each Day**

| Number of Humps | Number of Observations | Percent of Observations |
|---|---|---|
| **0 local minimum and maximum** | 1,100 | 14.26 |
| **1** | 2,416 | 31.32 |
| **2** | 1,883 | 24.41 |
| **3** | 871 | 11.29 |
| **4** | 658 | 8.53 |
| **5** | 325 | 4.21 |
| **6** | 116 | 1.50 |
| **7** | 205 | 2.66 |
| **8** | 70 | 0.91 |
| **9** | 61 | 0.79 |
| **10 or more** | 8 | 0.10 |

Kamakura Corporation, Deutsche Bundesbank, Bloomberg

The number of days with 0 or 1 humps (defined as the sum of local minima and maxima on that day's yield curve) was 45.58% of the total observations in the data set.

Finally, before proceeding, we count the number of occurrences of negative rates for each forward rate segment of the yield curve over the history provided by Bloomberg and report on the observed 91-day volatility of forward returns when the start of the period annualized forward rate is negative, zero, and positive.

Table V

Kamakura Corporation
HJM 14 Factor Model
German Bund
Using Daily Data from January 1, 1996 through March 27, 2017
Date of Analysis: July 12, 2017
Source: Kamakura Corporation, Deutsche Bundesbank, Bloomberg

| Count of Negative Quarterly Forward Rates | | | | | Standard Deviation of Change in Forwards | | | Observations for Standard Deviation | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Quarter Number | Observations | Negative | Zero | Positive | Negative | Zero | Positive | Total | Negative | Zero | Positive |
| 1 | 7713 | 841 | 35 | 6837 | 0.000193 | | 0.001038 | 7622 | 629 | 0 | 6993 |
| 2 | 7713 | 720 | 0 | 6993 | 0.000282 | | 0.001156 | 7622 | 826 | 0 | 6796 |
| 3 | 7713 | 914 | 0 | 6799 | 0.000258 | | 0.001229 | 7622 | 859 | 0 | 6763 |
| 4 | 7713 | 947 | 0 | 6766 | 0.000239 | | 0.001248 | 7622 | 832 | 0 | 6790 |
| 5 | 7713 | 921 | 0 | 6792 | 0.000230 | | 0.001253 | 7622 | 825 | 0 | 6797 |
| 6 | 7713 | 913 | 0 | 6800 | 0.000287 | | 0.001236 | 7622 | 686 | 0 | 6936 |
| 7 | 7713 | 776 | 0 | 6937 | 0.000324 | | 0.001209 | 7622 | 622 | 0 | 7000 |
| 8 | 7713 | 712 | 0 | 7001 | 0.000356 | | 0.001174 | 7622 | 582 | 0 | 7040 |
| 9 | 7713 | 672 | 0 | 7041 | 0.000384 | | 0.001145 | 7622 | 535 | 0 | 7087 |
| 10 | 7713 | 626 | 0 | 7087 | 0.000503 | | 0.001126 | 7622 | 479 | 0 | 7143 |
| 11 | 7713 | 572 | 0 | 7141 | 0.000516 | | 0.001142 | 7622 | 462 | 0 | 7160 |
| 12 | 7713 | 555 | 0 | 7158 | 0.000487 | | 0.001184 | 7622 | 436 | 0 | 7186 |
| 13 | 7713 | 529 | 0 | 7184 | 0.000545 | | 0.001212 | 7622 | 482 | 0 | 7140 |
| 14 | 7713 | 571 | 0 | 7142 | 0.000645 | | 0.001213 | 7622 | 479 | 0 | 7143 |
| 15 | 7713 | 545 | 0 | 7168 | 0.000736 | | 0.001211 | 7622 | 462 | 0 | 7160 |
| 16 | 7713 | 516 | 0 | 7197 | 0.000825 | | 0.001197 | 7622 | 403 | 0 | 7219 |
| 17 | 7713 | 422 | 0 | 7291 | 0.000889 | | 0.001150 | 7622 | 369 | 0 | 7253 |
| 18 | 7713 | 376 | 0 | 7337 | 0.000950 | | 0.001072 | 7622 | 320 | 0 | 7302 |
| 19 | 7713 | 321 | 0 | 7392 | 0.000915 | | 0.001021 | 7622 | 252 | 0 | 7370 |
| 20 | 7713 | 254 | 0 | 7459 | 0.000880 | | 0.001038 | 7622 | 224 | 0 | 7398 |
| 21 | 7713 | 226 | 0 | 7487 | 0.000880 | | 0.001076 | 7622 | 189 | 0 | 7433 |
| 22 | 7713 | 191 | 0 | 7522 | 0.000960 | | 0.001099 | 7622 | 131 | 0 | 7491 |
| 23 | 7713 | 133 | 0 | 7580 | 0.001026 | | 0.001097 | 7622 | 98 | 0 | 7524 |
| 24 | 7713 | 99 | 0 | 7614 | 0.001049 | | 0.001095 | 7622 | 57 | 0 | 7565 |
| 25 | 7713 | 57 | 0 | 7656 | 0.000194 | | 0.001093 | 7622 | 2 | 0 | 7620 |
| 26 | 7713 | 2 | 0 | 7711 | | | 0.001044 | 7622 | 1 | 0 | 7621 |
| 27 | 7713 | 1 | 0 | 7712 | 0.000303 | | 0.001011 | 7622 | 3 | 0 | 7619 |
| 28 | 7713 | 3 | 0 | 7710 | 0.000181 | | 0.001034 | 7622 | 7 | 0 | 7615 |
| 29 | 7713 | 7 | 0 | 7706 | 0.000089 | | 0.001082 | 7622 | 10 | 0 | 7612 |
| 30 | 7713 | 10 | 0 | 7703 | 0.000066 | | 0.001115 | 7622 | 4 | 0 | 7618 |
| 31 | 7713 | 4 | 0 | 7709 | 0.000067 | | 0.001128 | 7622 | 4 | 0 | 7618 |
| 32 | 7713 | 4 | 0 | 7709 | | | 0.001146 | 7622 | 0 | 0 | 7622 |

The smoothed Bloomberg data implies that annualized quarterly forward rates have been negative even more times than the Japanese Government Bond yield curve during the period studied. The 91-day standard deviation of forward rate changes for the negative observations ranged from nearly 2 basis points (0.0193%) on the short end of the curve to nearly 11 basis points (0.105%) at the 6-year point.

The same table for Japan shows that the volatility of forward rate changes can be calculated for the first forward rate on 303 observation dates when that forward rate was negative. The 91-day volatility was 0.018553%. For the 10,425 observation dates for which the first forward rate was positive in Japan, the volatility over 91 days was 0.135174%. For other forward rate maturities, the volatility of the negative rate observations gradually increased with maturity.

We emphasize two obvious points: rates can be and have been negative, and, when rates hit zero and below, interest rate volatility is not zero. It is positive but at a lower level than for positive forward rate observations. As a first approximation, we will use the observed volatility in Japan of 0.018553% as the floor for estimated interest rate volatility when the initial forward rate in that period was negative.

## III. Fitting Heath, Jarrow and Morton Parameters

A simple first step in constructing a multi-factor Heath, Jarrow and Morton model is to conduct principal components analysis on the forward rates that make up the relevant yield curve. For the German Bund market, at its longest maturity, these quarterly segments consist of one three-month spot rate and 119 forward rates. Over 7,622 observations, the principal components analysis indicates in Table VI that the first factor explains only 57.73% of the movement in forward rates over the full curve. For a high degree of explanatory power, the principal components analysis indicates that 17 to 18 factors will be necessary.

**Table VI**

```
Principal components/correlation              Number of obs    =      7,622
                                              Number of comp.  =         23
                                              Trace            =        119
    Rotation: (unrotated = principal)         Rho              =     1.0000
```

| Component | Eigenvalue | Difference | Proportion | Cumulative |
|---|---|---|---|---|
| Comp1 | 68.7041 | 53.2737 | 0.5773 | 0.5773 |
| Comp2 | 15.4303 | 3.88883 | 0.1297 | 0.7070 |
| Comp3 | 11.5415 | 1.08111 | 0.0970 | 0.8040 |
| Comp4 | 10.4604 | 7.02365 | 0.0879 | 0.8919 |
| Comp5 | 3.43674 | .614089 | 0.0289 | 0.9208 |
| Comp6 | 2.82265 | 1.18805 | 0.0237 | 0.9445 |
| Comp7 | 1.6346 | .517056 | 0.0137 | 0.9582 |
| Comp8 | 1.11755 | .188401 | 0.0094 | 0.9676 |
| Comp9 | .929144 | .0507911 | 0.0078 | 0.9754 |
| Comp10 | .878353 | .163094 | 0.0074 | 0.9828 |
| Comp11 | .715259 | .321209 | 0.0060 | 0.9888 |
| Comp12 | .39405 | .0577644 | 0.0033 | 0.9921 |
| Comp13 | .336286 | .085519 | 0.0028 | 0.9950 |
| Comp14 | .250767 | .0649611 | 0.0021 | 0.9971 |
| Comp15 | .185806 | .08445 | 0.0016 | 0.9986 |
| Comp16 | .101356 | .0587625 | 0.0009 | 0.9995 |
| Comp17 | .0425934 | .0294661 | 0.0004 | 0.9998 |
| Comp18 | .0131273 | .00932095 | 0.0001 | 1.0000 |
| Comp19 | .00380635 | .00281308 | 0.0000 | 1.0000 |
| Comp20 | .00099327 | .0005502 | 0.0000 | 1.0000 |
| Comp21 | .00044307 | .00034545 | 0.0000 | 1.0000 |
| Comp22 | .0000976204 | .0000316971 | 0.0000 | 1.0000 |
| Comp23 | .0000659232 | .0000573349 | 0.0000 | 1.0000 |
| Comp24 | 8.58833e-06 | 3.02667e-06 | 0.0000 | 1.0000 |
| Comp25 | 5.56166e-06 | 4.34282e-06 | 0.0000 | 1.0000 |
| Comp26 | 1.21884e-06 | 1.21884e-06 | 0.0000 | 1.0000 |
| Comp27 | 0 | 0 | 0.0000 | 1.0000 |

With this analysis as background, we begin the Heath, Jarrow and Morton fitting process.

In the studies done so far, the number of statistically significant factors is summarized below:

| | | |
|---|---|---|
| Australia: | Commonwealth Government Securities, | 14 factors |
| Canada: | Government of Canada Securities, | 12 factors |
| Germany*: | Bunds, | 14 factors |
| Japan: | Japanese Government Bonds | 8 factors |
| Singapore: | Singapore Government Securities | 9 factors |
| Spain | Spanish Government Securities | 11 factors |
| Sweden: | Swedish Government Securities | 11 factors |
| Thailand | Thai Government Securities | 11 factors |
| United Kingdom | United Kingdom Government Securities | 14 factors |
| United States: | Treasury Securities | 10 Factors |

* Prior version

Note that our prior term structure model fitting exercise for the German Bund market resulted in 14 statistically significant factors through June 30, 2015.

We now fit a multi-factor Heath, Jarrow and Morton model to German Bund zero-coupon yield data from January 1, 1996 to March 27, 2017. Germany is fairly unique in that only one data regime was experienced over the historical period.

The procedures used to derive the parameters of a Heath, Jarrow and Morton model are described in detail in Jarrow and van Deventer (June 16, 2015 and May 5, 2017).

We followed these steps to estimate the parameters of the model:

- We extract the zero-coupon yields and zero-coupon bond prices for all quarterly maturities out to 30 years for all daily observations for which the 30-year coupon-bearing bond yield is available (i.e. all observations). This is done using Kamakura Risk Manager, version 8.1, using the maximum smoothness forward rate approach to fill the quarterly maturity gaps in the zero-coupon bond data.
- We use overlapping 91-day intervals to measure changes in forward rates, avoiding the use of “quarterly” data because of the unequal lengths of calendar quarters. Because overlapping observations trigger auto-correlation, “HAC” (heteroscedasticity and autocorrelation consistent) standard errors are used. The methodology is that of Newey-West with 91-day lags.
- We consider 14 potential explanatory factors: the idiosyncratic portion of the movements in quarterly forward rates that mature in 6 months, 1 year, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, and 30 years. Ten factors are required by the Bank for International Settlements market risk guidelines published in January 2016 and relevant to the Fundamental Review of the Trading Book.
- We calculate the discrete changes in forward returns as described in the parameter technical guide. Because the discrete changes are non-linear in the no-arbitrage framework of Heath, Jarrow and Morton, we use non-linear least squares to fit interest rate volatility.
- We use a different non-linear regression for each segment of the yield curve. We considered a panel-based approach, but we rejected it for two reasons: first, the movement of parameters as maturity lengthens is complex and not easily predictable before estimation; second, the residual unexplained error in forward rates is very, very small, so the incremental merit of the panel approach is minimal.
- We then begin the process of creating the orthogonalized risk factors that drive interest rates using the Gram-Schmidt procedure. These factors are assumed to be uncorrelated independent random variables that have a normal distribution with mean zero and standard deviation of 1.
- Because interest volatility is assumed to be stochastic, simulated out-of-sample forward rates will not in general be normally distributed. We also calculate constant volatility parameters and choose the most accurate from the constant volatility and stochastic volatility models estimated.
- In the estimation process, we added factors to the model as long as each new factor provided incremental explanatory power. The standard suite of models in both cases includes 1 factor, 2 factors, 3 factors, 6 factors and “all factors,” which varies by country.

We postulate that interest rate volatility for risk factor j at each forward rate maturity k is a cubic function of the annualized forward rate that prevails for the relevant risk factor j at the beginning of each 91-day period:

$$\sigma_{jk} = \max[b_{0,jk}, b_{0,jk} + b_{1,jk}f + b_{2,jk}f^2 + b_{3,jk}f^3] \; if \; f > 0,$$
$$\sigma_{jk} = b_{0,jk} \; if \; f \leq 0,$$

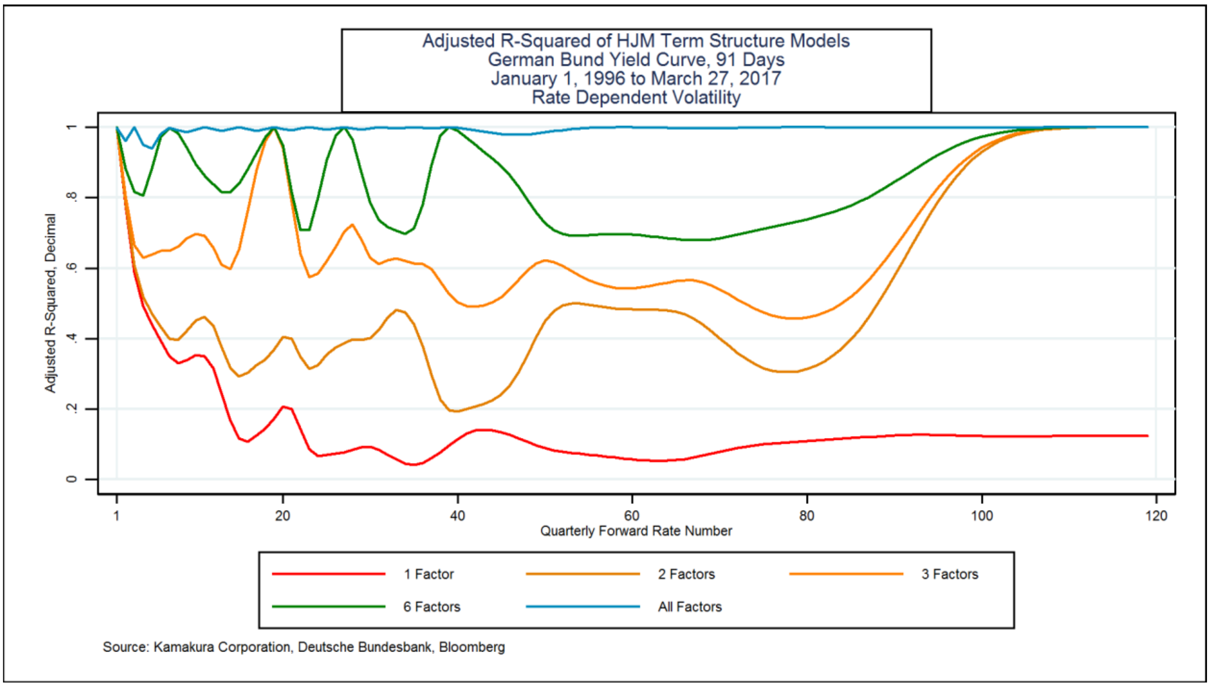
When the initial forward rate is negative, we postulate that interest rate volatility is a constant. Because of the Japan volatility data reported above, we expect $b_{0,jk}$ to be close to 0.018553%.

We use the resulting parameters and accuracy tests to address the hypothesis that a one factor model is "good enough" for modeling German Bund yields in the Appendix. We report the accuracy results for 1, 2, 3, 6 and all (14) factors. The factors are the idiosyncratic variation in quarterly forward rates at each of 14 maturities. The factors, described by the maturity of the quarterly forward rate used, are added to the model in this order:

Factor 201, 0.5 years
Factor 202, 30 years
Factor 203, 5 years
Factor 204, 10 years
Factor 205, 2 years
Factor 206, 7 years
Factor 207, 15 years
Factor 208, 1 year
Factor 209, 20 years
Factor 210, 8 years
Factor 211, 3 years
Factor 212, 9 years
Factor 213, 4 years
Factor 214, 6 years

Exhibit VII summarizes the adjusted r-squared for the non-linear equations for each of the 119 quarterly forward rate segments that make up the German Bund yield curve:

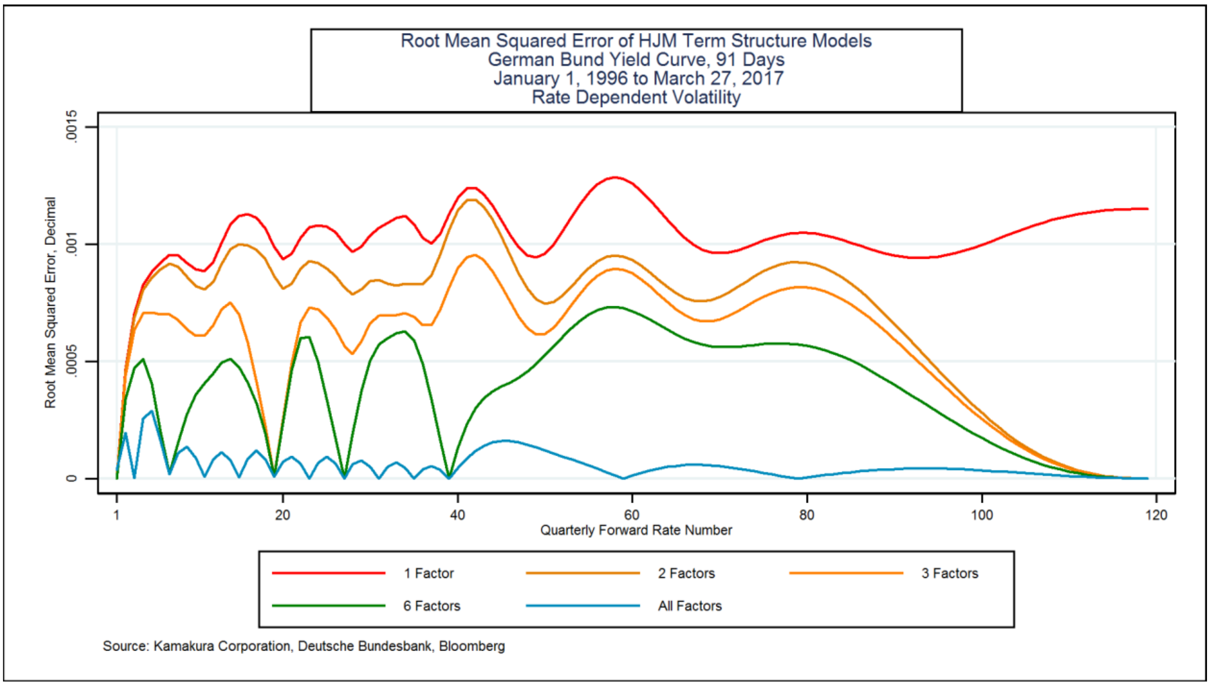
## Exhibit VII




The adjusted r-squared for the best practice model over each of the forward rates is plotted in blue and is near 100% for all 119 quarterly segments of the yield curve. The one factor model in red, by contrast, does a poor job of fitting 91-day movements in the quarterly forward rates. The adjusted r-squared is good, of course, for the first forward rate since the short rate is the standard risk factor in a one factor term structure model. Beyond the first quarter, however, explanatory power declines rapidly. The adjusted r-squared of the one factor model never exceeds 20% after the first 21 quarterly forward rates and is below that level at most maturities.

The root mean squared error for the 1, 2, 3, 6 and all (14) factor stochastic volatility model is shown in Exhibit VIII.

**Exhibit VIII**




The root mean squared error for the 14-factor model is less than 0.015% at every maturity along the yield curve once the maturity exceeds 2 years (8 quarters). This result should not come as a surprise to a serious analyst, because it is very similar to the results of the best practice Heath, Jarrow and Morton term structure models for Japanese Government Bonds, Government of Canada Bonds, Australian Commonwealth Government Securities, Singapore Government Securities, Spanish Government Securities, Swedish Government Securities, United Kingdom Government Securities and Thai Government Bond yields.

## Validation of Orthogonolization

Exhibit IX provides verification that the Gram-Schmidt procedure produced historical correlations that were nearly zero. Some correlation remains because we are using 91-day overlapping observations.

## Exhibit IX

| | Fact~201 | Fact~202 | Fact~203 | Fact~204 | Fact~205 | Fact~206 | Fact~207 |
|---|---|---|---|---|---|---|---|
| Factor201 | 1.0000 | | | | | | |
| Factor202 | -0.0104 | 1.0000 | | | | | |
| Factor203 | 0.0035 | -0.0191 | 1.0000 | | | | |
| Factor204 | -0.0099 | 0.0171 | -0.0074 | 1.0000 | | | |
| Factor205 | 0.0125 | -0.0623* | -0.0353* | -0.0382* | 1.0000 | | |
| Factor206 | -0.0012 | 0.0105 | 0.0004 | -0.0078 | -0.0085 | 1.0000 | |
| Factor207 | -0.0135 | 0.0228* | -0.0015 | -0.0009 | -0.0212 | 0.0048 | 1.0000 |
| Factor208 | 0.0145 | -0.0034 | -0.0228* | -0.0094 | -0.0166 | -0.0086 | -0.0326* |
| Factor209 | 0.0034 | -0.0101 | -0.0006 | -0.0026 | -0.0021 | 0.0017 | 0.0021 |
| Factor210 | -0.0008 | -0.0069 | 0.0063 | -0.0019 | -0.0097 | 0.0075 | -0.0014 |
| Factor211 | 0.0224 | -0.0223 | -0.0196 | -0.0014 | 0.0156 | -0.0076 | -0.0123 |
| Factor212 | 0.0024 | -0.0058 | -0.0036 | -0.0042 | -0.0032 | -0.0027 | 0.0011 |
| Factor213 | -0.0002 | -0.0563* | 0.0571* | -0.0217 | 0.0369* | -0.0030 | 0.0023 |
| Factor214 | -0.0022 | 0.0128 | 0.0098 | -0.0067 | -0.0022 | -0.0054 | -0.0010 |

| | Fact~208 | Fact~209 | Fact~210 | Fact~211 | Fact~212 | Fact~213 | Fact~214 |
|---|---|---|---|---|---|---|---|
| Factor208 | 1.0000 | | | | | | |
| Factor209 | 0.0022 | 1.0000 | | | | | |
| Factor210 | -0.0011 | -0.0002 | 1.0000 | | | | |
| Factor211 | 0.0274* | -0.0211 | -0.0163 | 1.0000 | | | |
| Factor212 | 0.0065 | 0.0064 | -0.0028 | -0.0064 | 1.0000 | | |
| Factor213 | -0.0119 | -0.0111 | -0.0000 | -0.0112 | 0.0133 | 1.0000 | |
| Factor214 | -0.0093 | 0.0060 | 0.0018 | -0.0185 | 0.0011 | 0.0048 | 1.0000 |

There are 14(13)/2 = 91 correlations, 10 of which are different from zero in a statistically significant way due to the use of overlapping observations. If we recalculated the correlations using 1 data set for each group of non-overlapping observations, we would see a lack of statistically significant correlation approximately 95% of the time.

## Bayesian Considerations in Model Validation

Kamakura term structure model validation is conducted in the spirit of Bayesian iterative model building as outlined by Gelman et al. This quote[4] from Gelman et al [2013] explains the Bayesian estimation process:

“The process of Bayesian data analysis can be idealized by dividing it into the following three steps:

1. Setting up a full probability model—a joint probability distribution for all observable and unobservable quantities in a problem. The model should be consistent with knowledge about the underlying scientific problem and the data collection process.

[4] Gelman et all [2013], page 3.

2. Conditioning on the observed data: calculating and interpreting the appropriate posterior distribution—the conditional probability distribution of the unobserved quantities of ultimate interest, given the observed data.

3. Evaluating the fit of the model and the implications of the resulting posterior distribution: how well does the model fit the data, are the substantive conclusions reasonable, and how sensitive are the results to the modeling assumptions in step 1? In response, one can alter or expand the model and repeat the three steps."

Jarrow and van Deventer go on to explain that the iterative process described above by Gelman et al is especially important in fitting Heath, Jarrow and Morton parameters for the following reasons:

a. Negative interest rates have been observed in Japan, Hong Kong and many European countries, but many other countries, including the U.S., have yet to experience negative rates. In the U.S. case, Bloomberg notes on its website that it overrides observed negative yields in the market with zero values.
b. The "knowledge about the underlying scientific problem" from the historical data available is as follows: (1) negative rates are possible, (2) they are much less likely to occur than positive rates, (3) interest rate volatility that results when rates are negative is of high interest but the historical data is either limited or non-existent, depending on the country, and (4) an international data set would best shed light on this and other HJM issues.

There are other issues relevant to estimation:

c. As noted by Heath, Jarrow and Morton [1992], stochastic volatility driven by interest rate levels must be capped to avoid a positive probability of infinitely high interest rates
d. Subject to this cap, most market participants expect interest rate volatility to rise as rates rise and that the interest rate volatility that prevails when rates are negative represents the lowest level of volatility that would prevail. Historical experience with negative rates so far around the world makes it clear that interest rate volatility does not go to zero at any rate level.
e. Most market participants believe that the empirical drift in forward rates that occurs (i.e. the change in observed empirical interest rates in the case where all interest rate shocks are zero) varies by the level of interest rates. The stochastic volatility model described in this paper assumes that empirical drift is a cubic function of annualized forward rates.

To summarize, a model validation effort in the Bayesian spirit would address at least these issues:

- Tests of smoothness of simulated curves
- Tests to confirm existence of negative rates in selected circumstances in the simulation
- Comparison of simulated risk neutral and empirical yields

- Time series distribution of simulated risk neutral and empirical yields

We first measure the sensitivity of interest rate volatility to shifts in a flat forward rate curve from 0% to 7% in 1% increments. As one would expect from a Bayesian "scientific knowledge" point of view, an increase in a flat forward rate curve increases the size of forward rate changes due to changes in the first risk factor:

**Exhibit X**

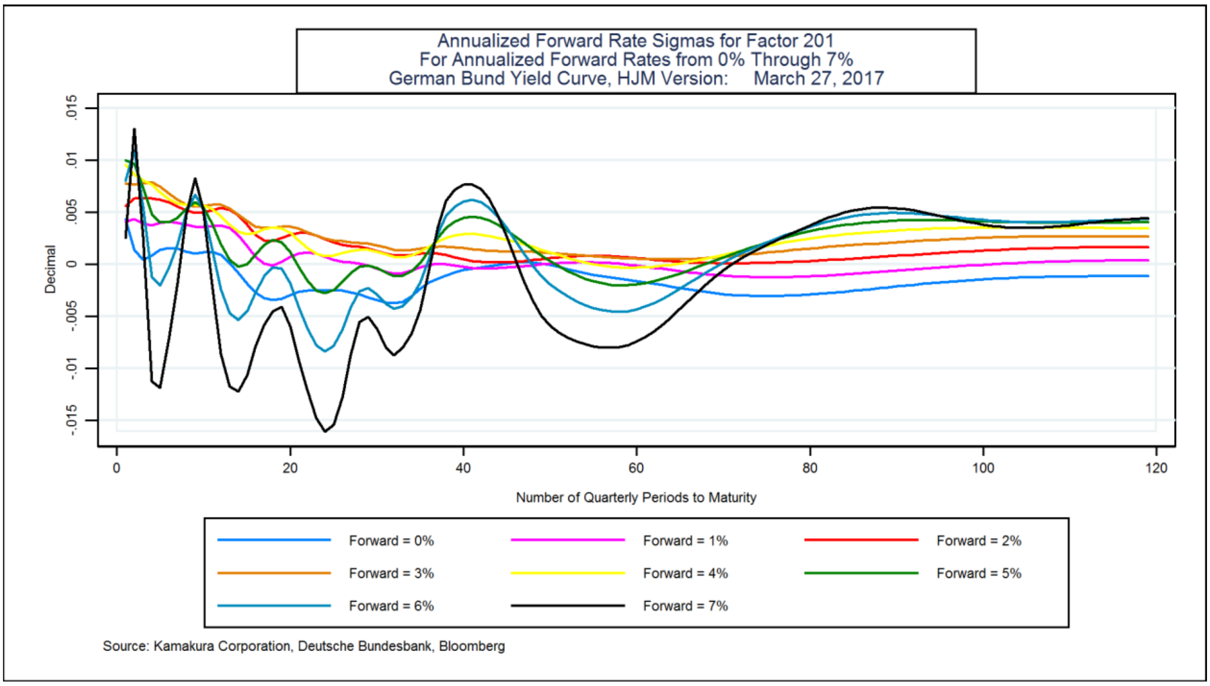
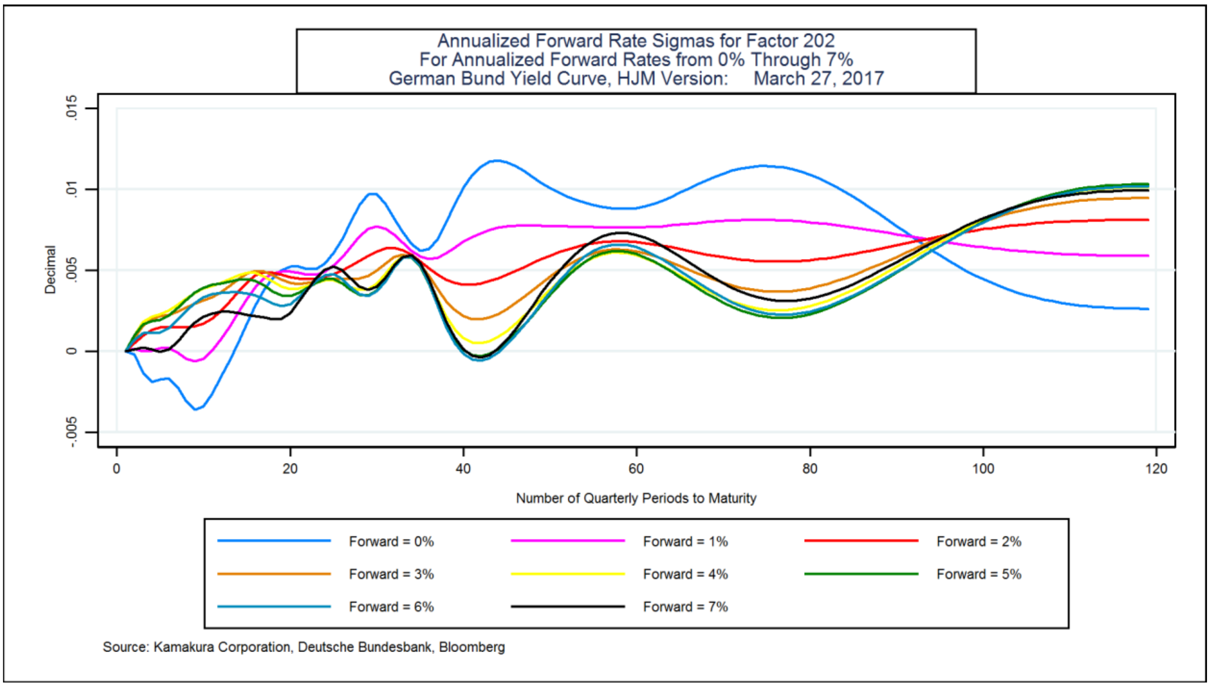



Exhibit XI shows the impact of the same shifts in a flat forward rate curve on interest rate volatility for the 2nd factor, the idiosyncratic movements in the quarterly forward rate maturing in 30 years:

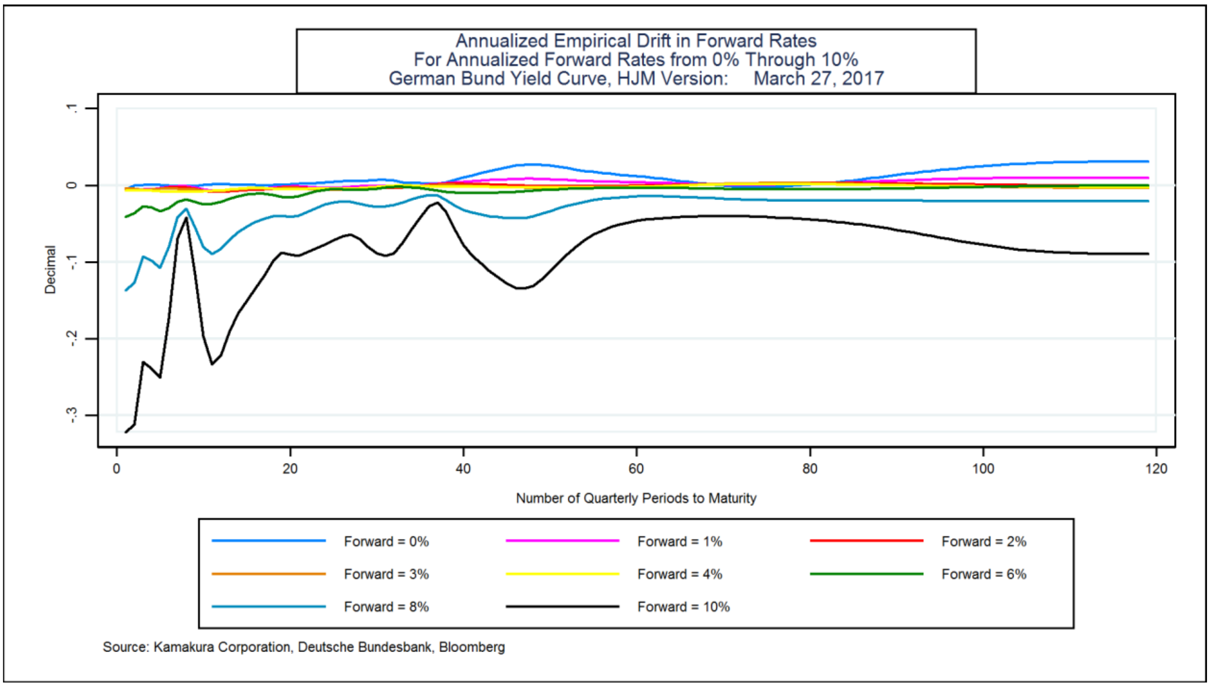
Exhibit XI




As rates rise, volatility shifts in a complex but smooth way that reflects historical experience.

Under the Heath, Jarrow and Morton no-arbitrage constraints, the risk neutral drift in forward rates is completely determined by the term structure of interest rate volatilities. The empirical drifts, however, can only be determined econometrically. We again assume a cubic relationship that is a function of the annualized forward rate for that maturity. When we shift a flat forward rate curve upward from 0%, we get the results in Exhibit XII for the changes in empirical drift:

**Exhibit XII**




In general, a rise in the initial forward rate curve decreases the drift in forward rates until it becomes negative for all maturities. This is an econometric confirmation of the mean reversion phenomenon.

### A. A Forward-Looking Simulation to Generate the "Posterior Distribution" of Forward Rates

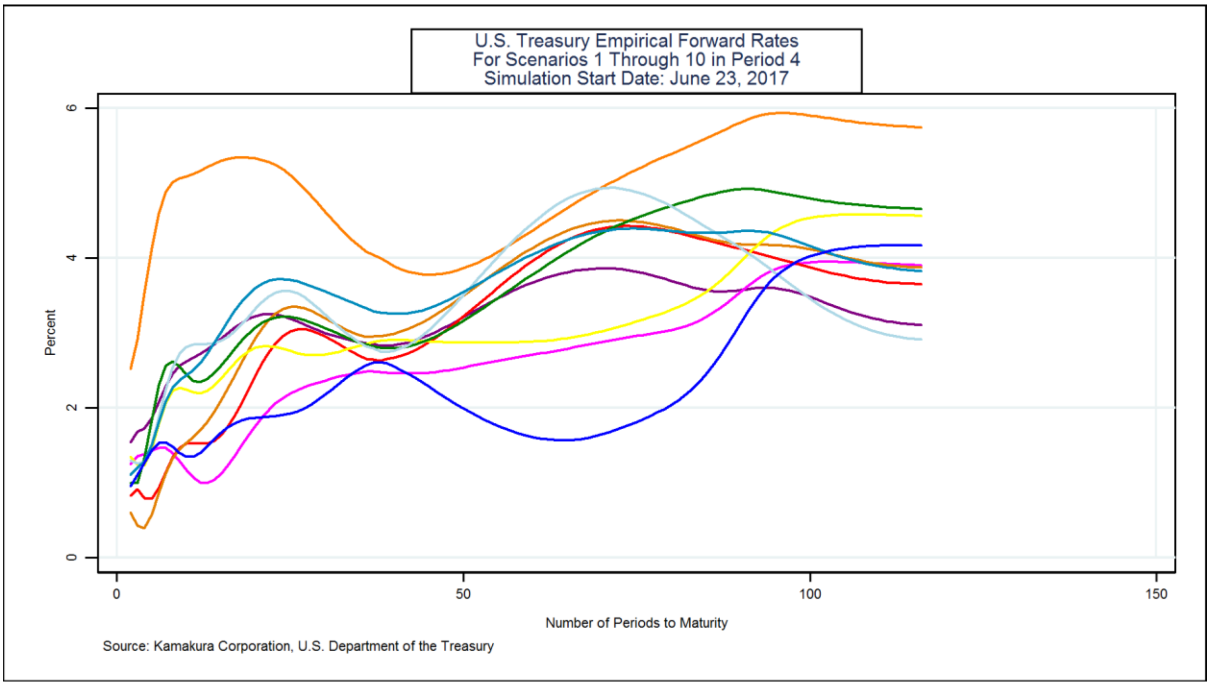
We now conduct a sample inspection of these issues with the aid of a forward-looking out-of-sample simulation of bond yields. For this version of the paper, we use U.S. data to illustrate the process. In a forthcoming version of the paper, the U.S. simulation will be replaced with a simulation based on Bund yields. The example has the following specifications:

- Yield curve: U.S. Treasury yields
- Date of yields: June 30, 2017
- Number of scenarios: 250,000
- Simulation time horizon: 30 years
- Simulation periodicity: Quarterly

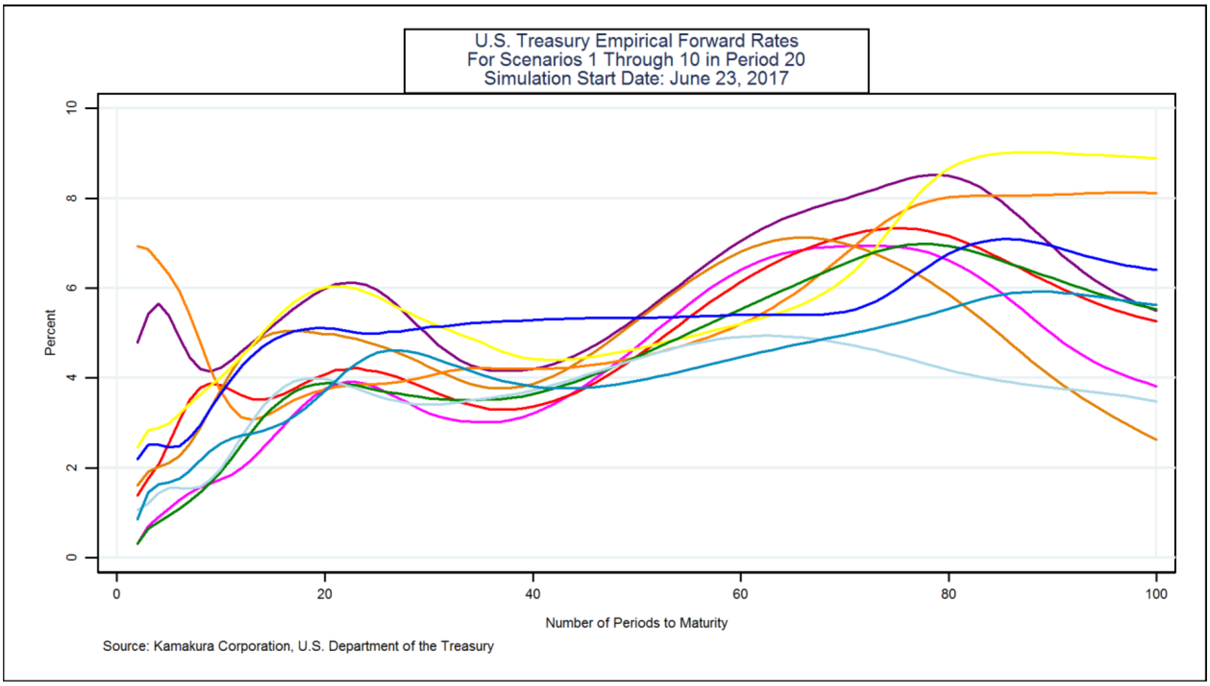
### B. Smoothness Validation

First, we select a random sample of 10 scenarios at each time step and visually examine them for smoothness. We can also use the discrete formula for smoothness given above to identify any outliers and examine the scenarios in question.

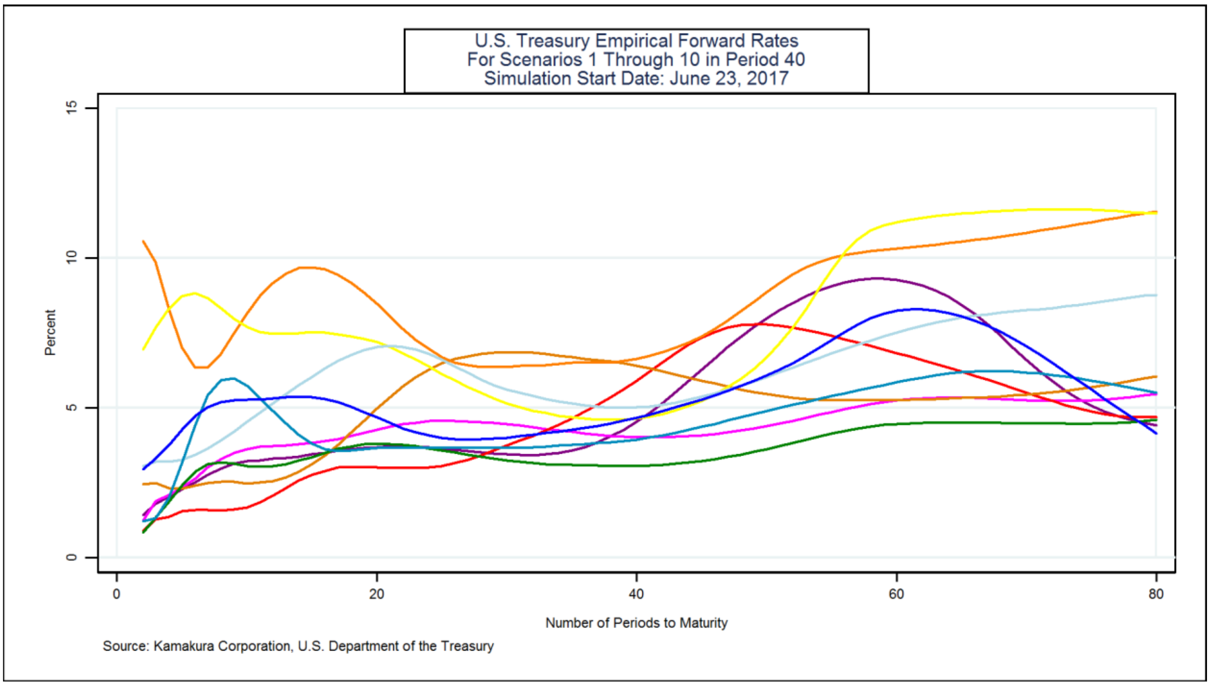
## Exhibit XIII: 1 year



## Exhibit XIV: 5 years



## Exhibit XV: 10 years



## Exhibit XVI: 20 years

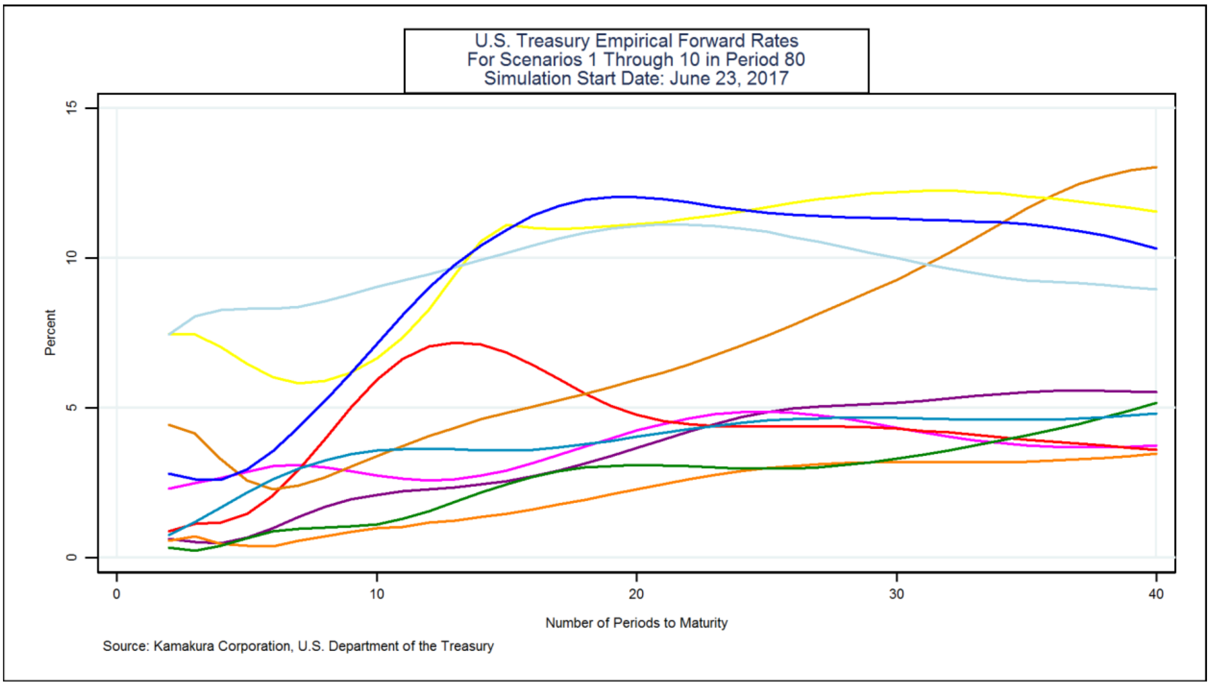
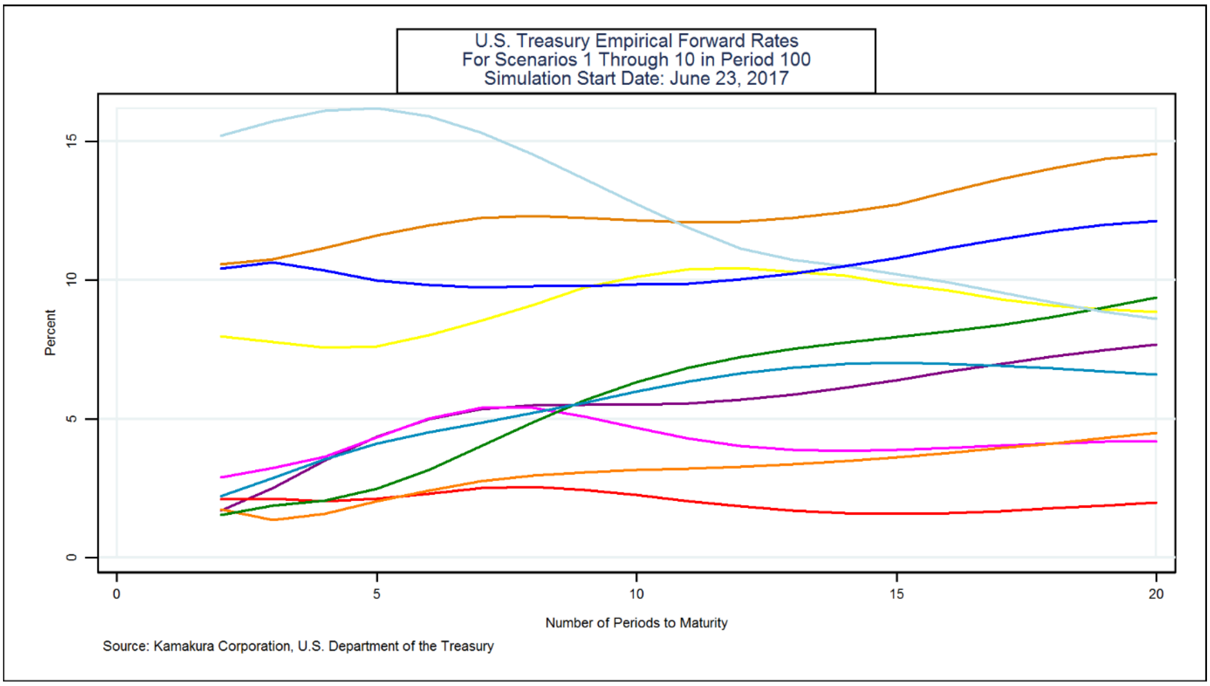


**Exhibit XVII: 25 years**

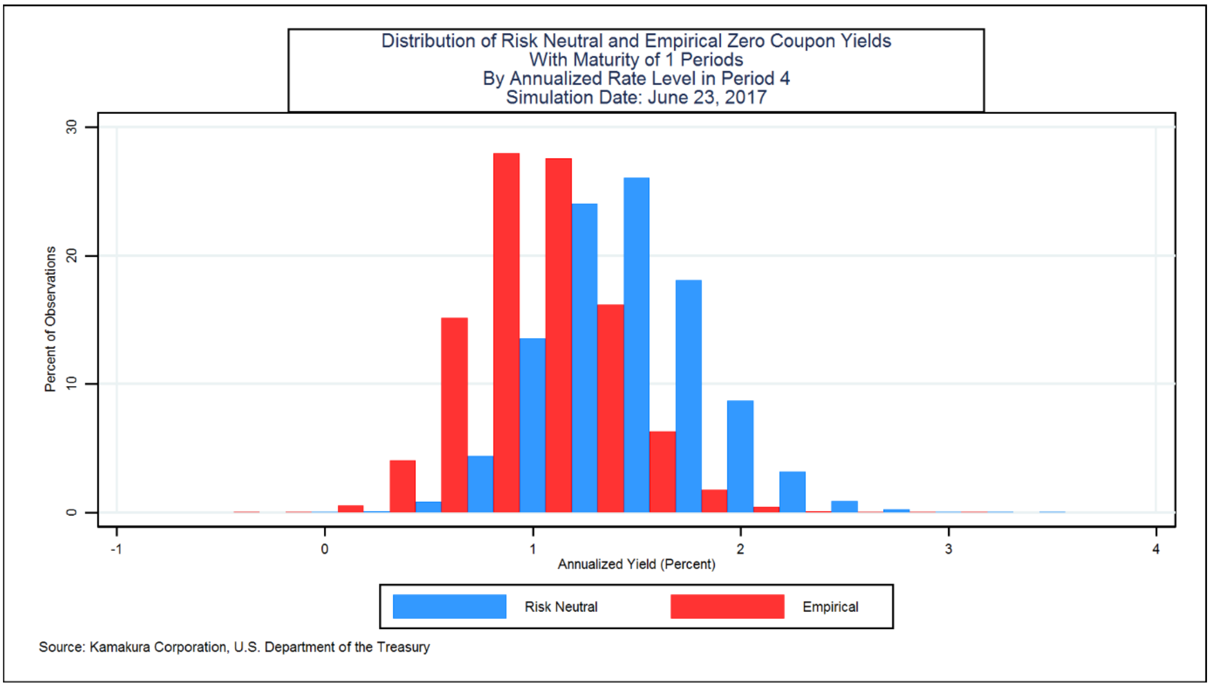



These graphs provide informal confirmation that nothing in the model estimation procedure has introduced artificial lumpiness in the simulated yield curves. A quantitative assessment of the smoothness of all 250,000 yield curves at each time step provides the more formal confirmation that the yield curves simulated are realistically smooth.

## C. Distribution of Simulated Risk Neutral and Empirical Rates

We now examine the probability distributions of risk neutral and empirical simulated rates at various maturities over time. We seek to detect visually any points in time at which the simulated distribution of yields is strange or unrealistic.

### Exhibit XVIII: Three Month Yields at 1 Year



## Exhibit XIX: Three Month Yields at 5 Years

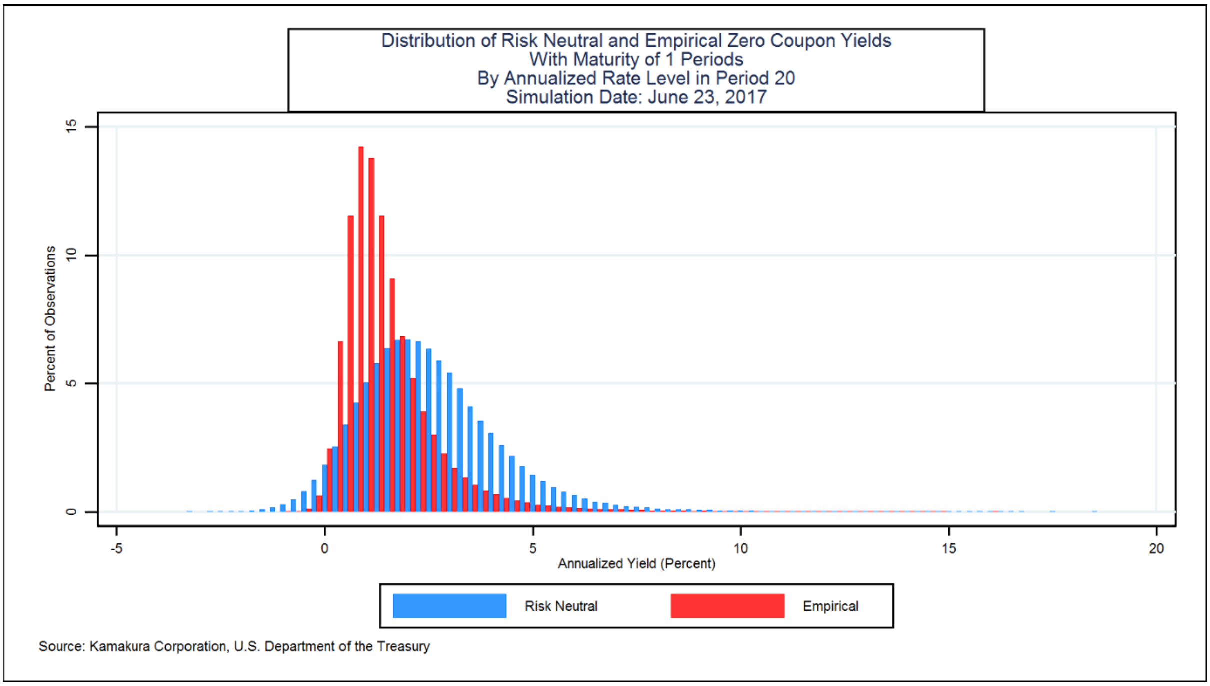
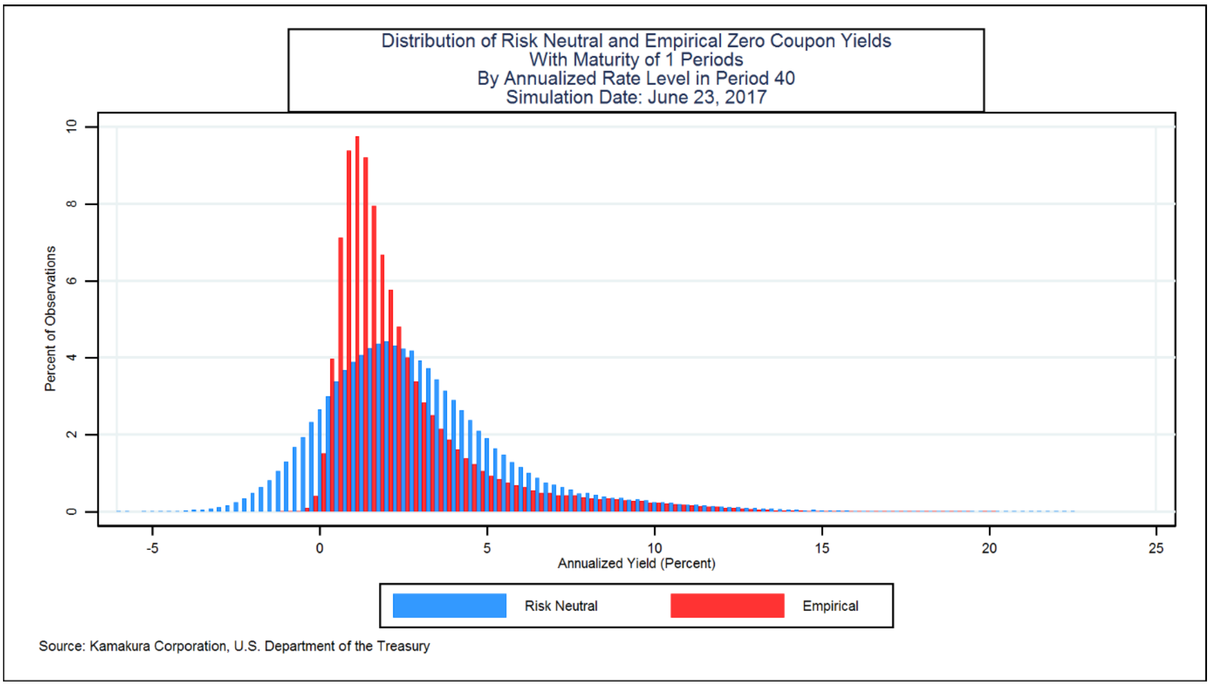


**Exhibit XX: Three Month Yields at 10 Years**



## Exhibit XXI: Three Month Yields at 20 Years

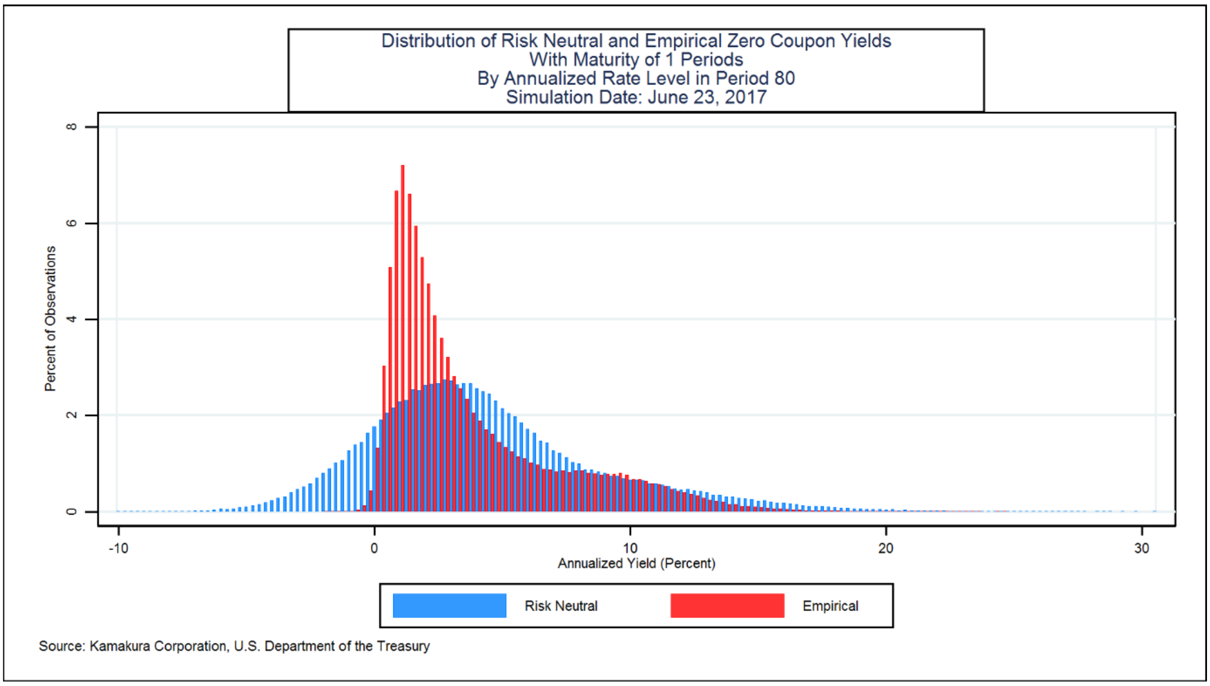
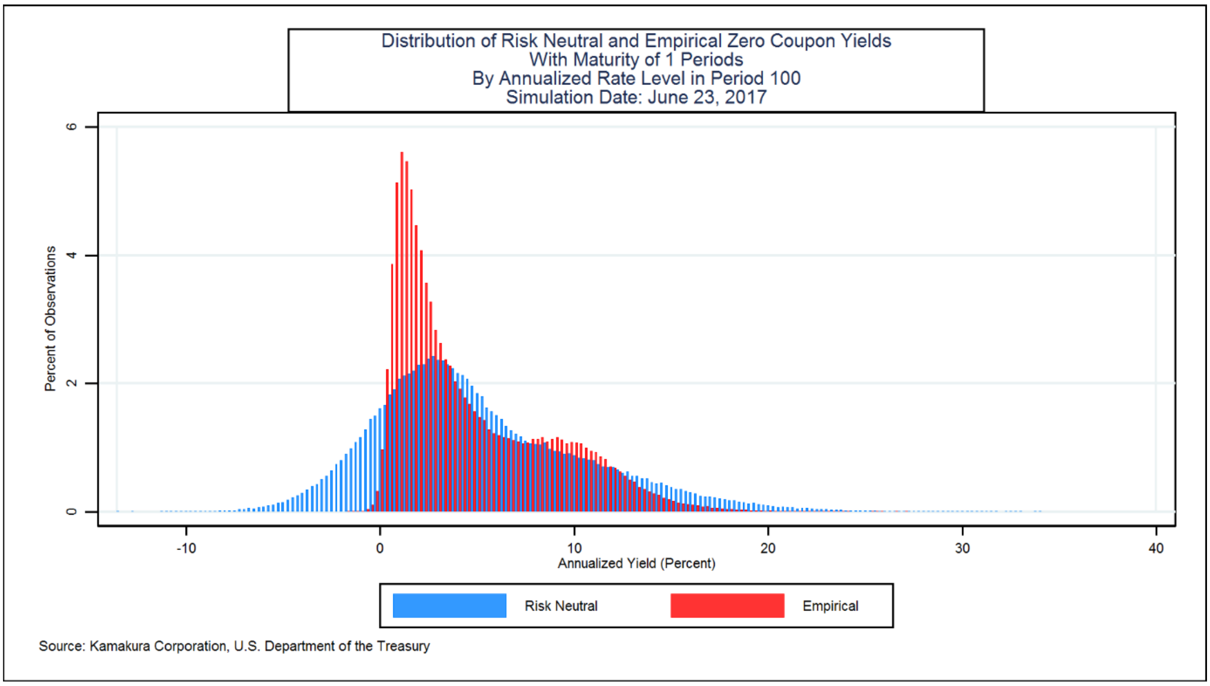
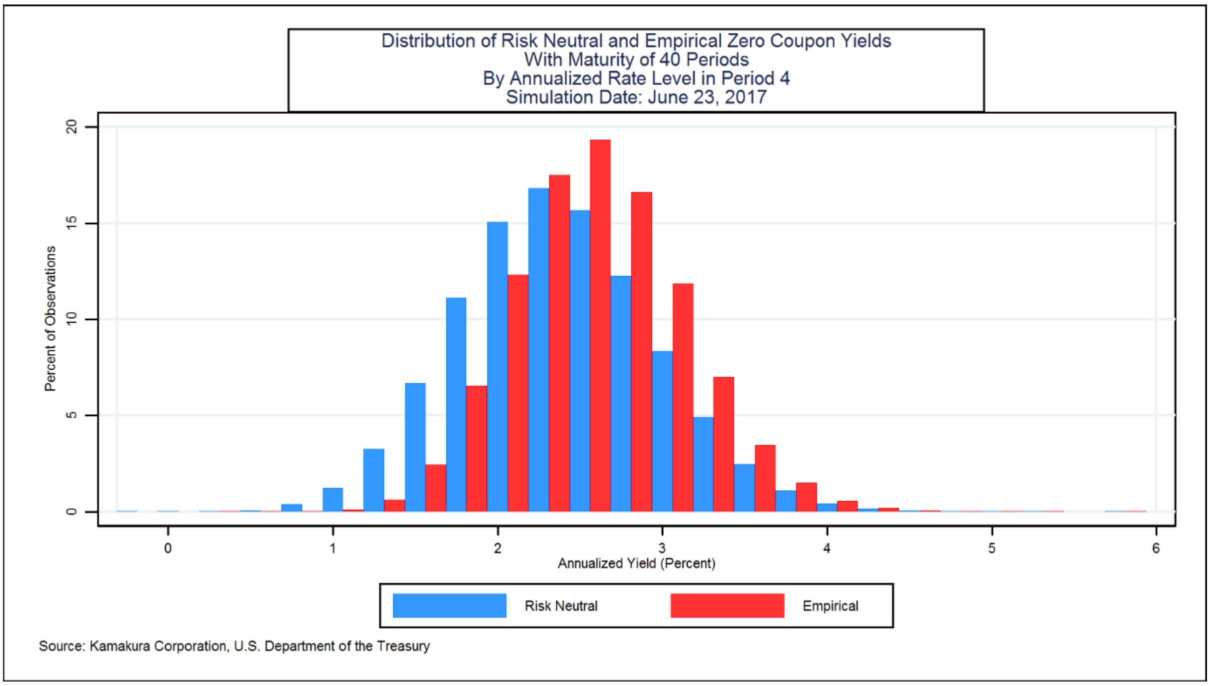


**Exhibit XXII: Three Month Yields at 25 Years**



## Exhibit XXIII: 10 Year Yields at 1 Year



## Exhibit XXIV: 10 Year Yields at 5 Years

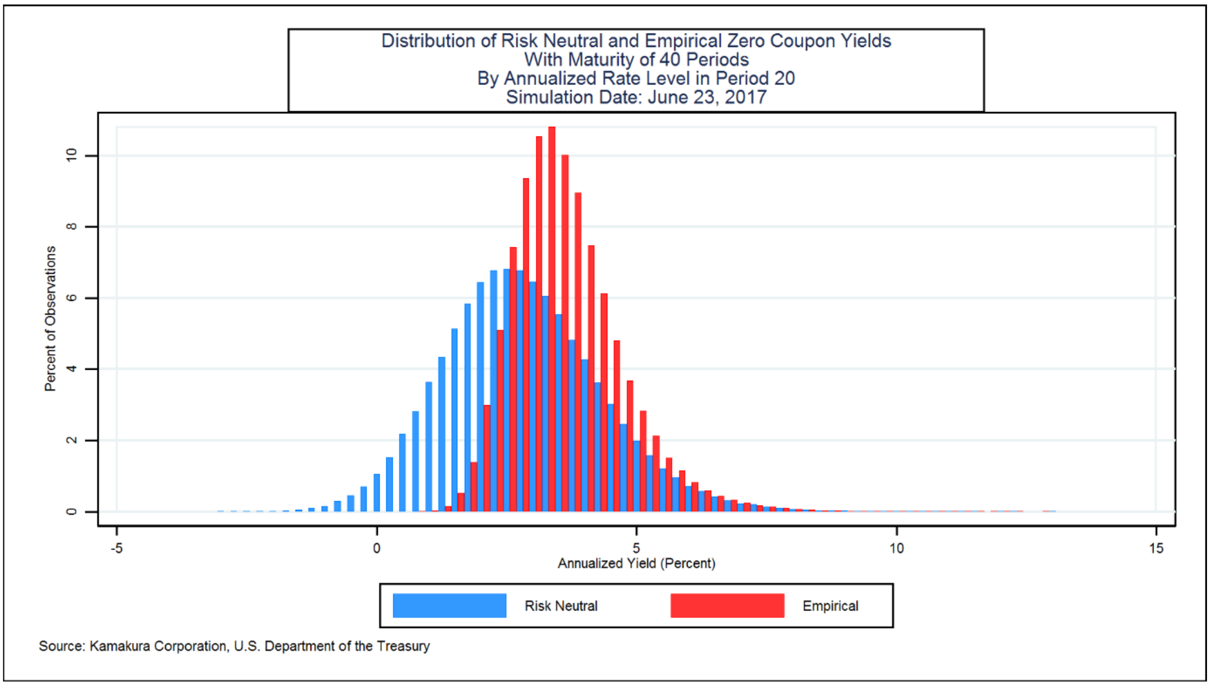
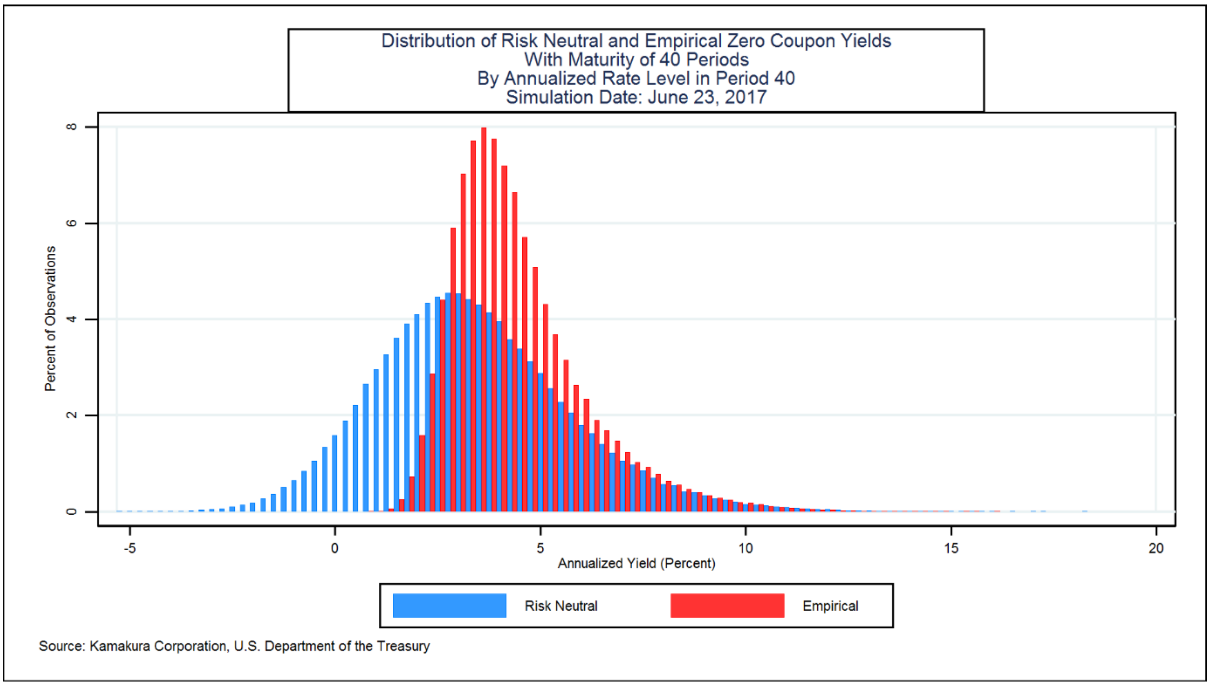
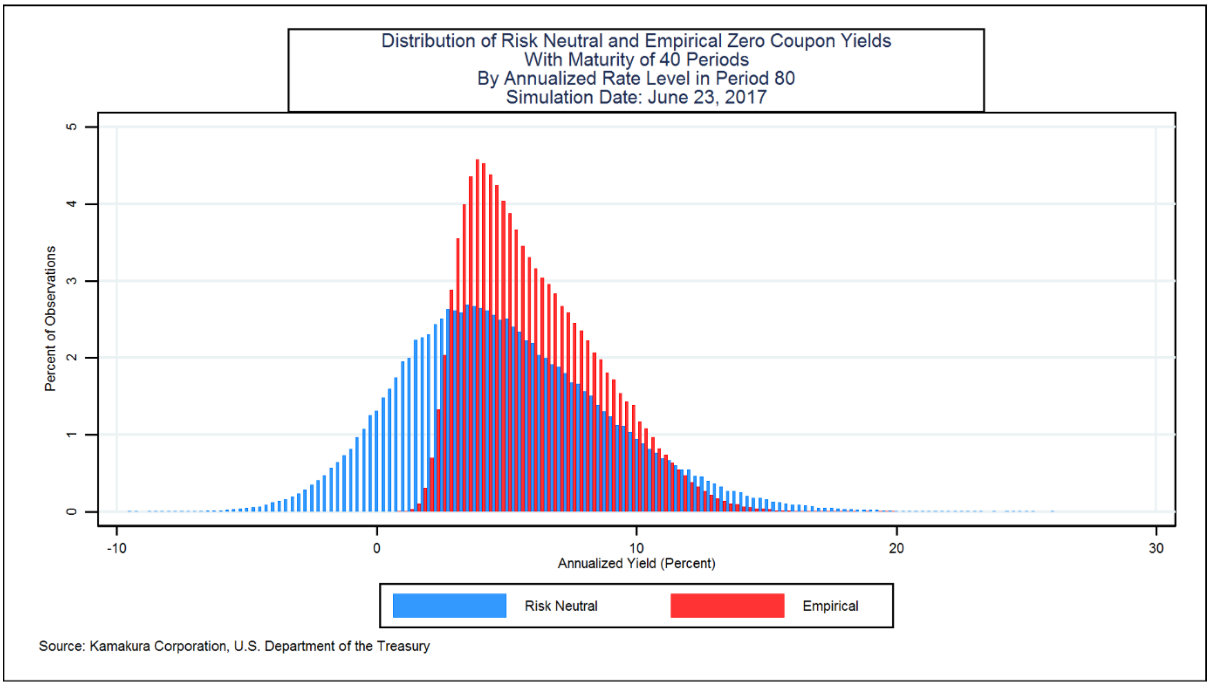


**Exhibit XXV: 10 Year Yields at 10 Years**



### Exhibit XXVI: 10 Year Yields at 20 Years

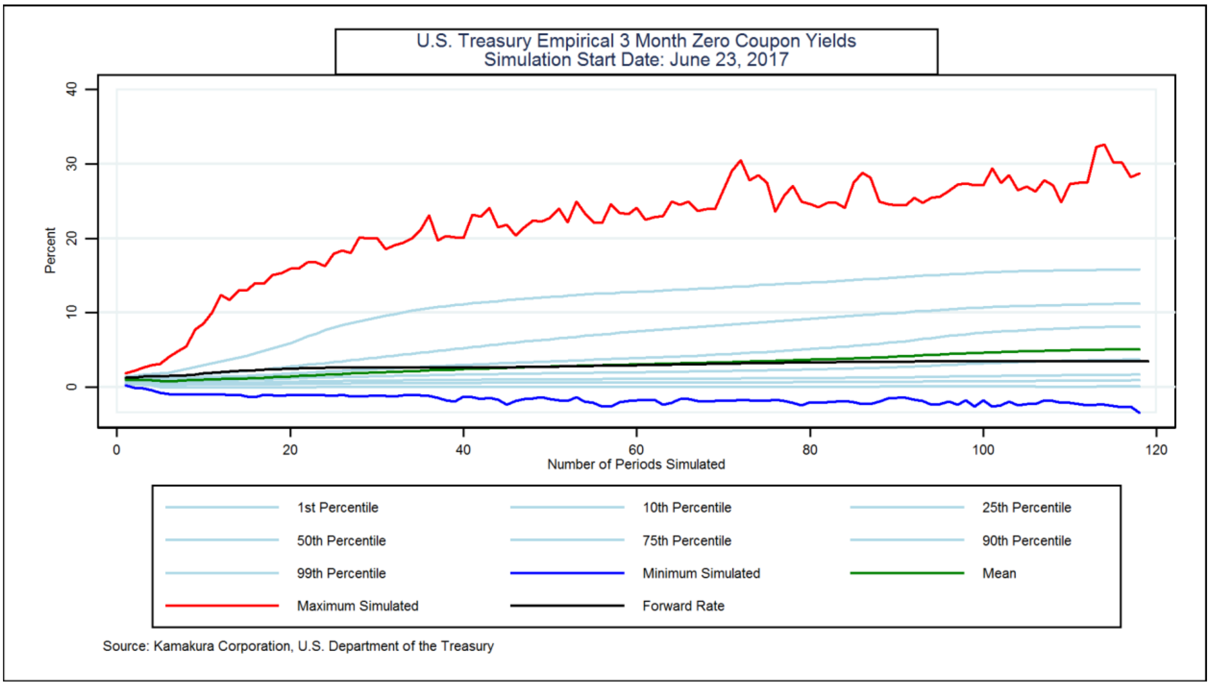



We conclude that the simulation is reasonable from multiple dimensions. Rates can be negative but (for empirical yields) the probability of negative rates is low. On the high end of the spectrum, rates do rise to the 20% range but with a very low probability.

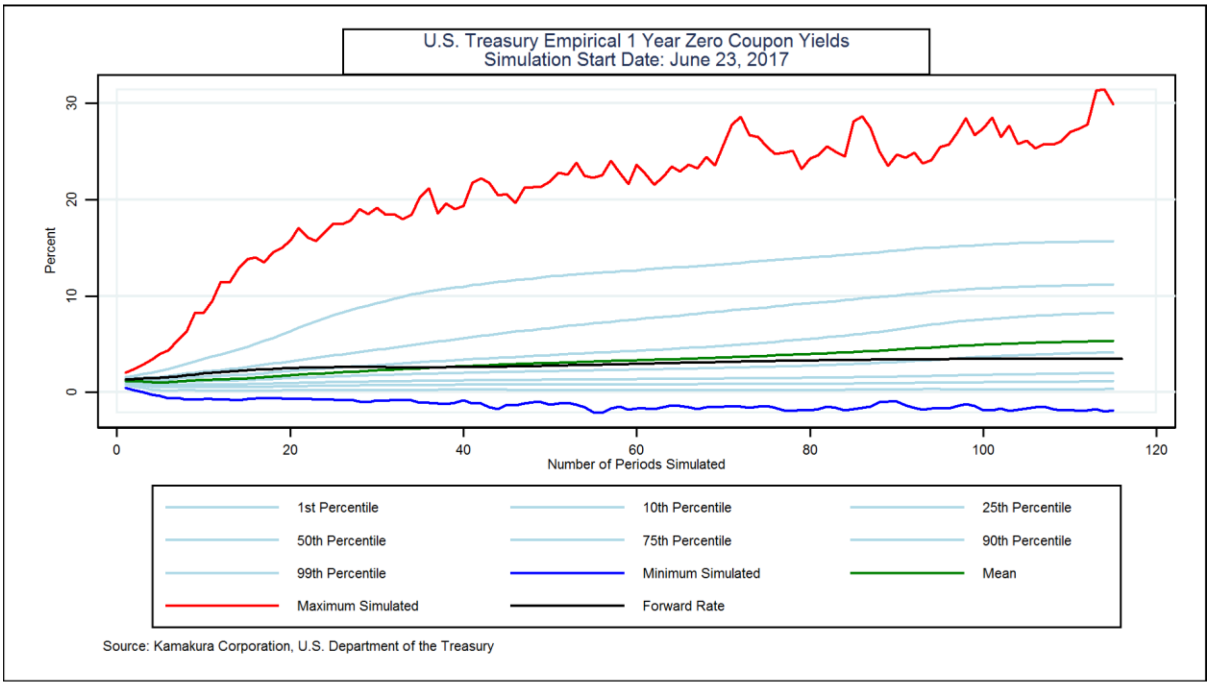
### D. Time Series Distribution of Simulated Yields

We now plot the time series graphs of the mean, median, high, low and various percentiles for empirical rates over the 30-year horizon of the simulation.

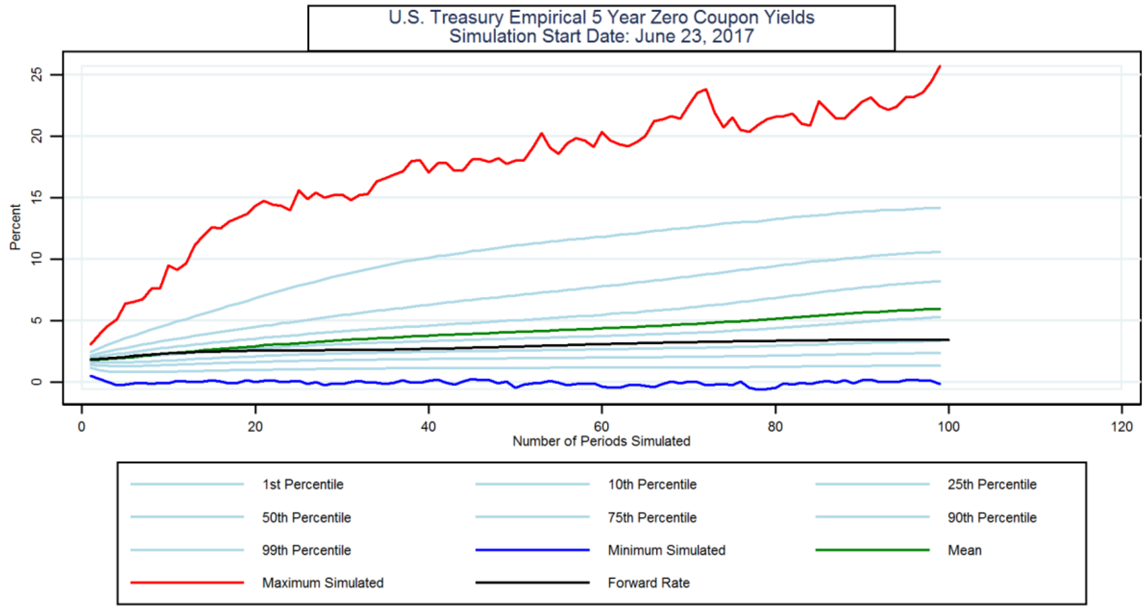
### Exhibit XXVII: 3 Month Yields



## Exhibit XXVIII: 1 Year Yields



## Exhibit XXIX: 5 Year Yields




Source: Kamakura Corporation, U.S. Department of the Treasury

## Exhibit XXX: 10 Year Yields

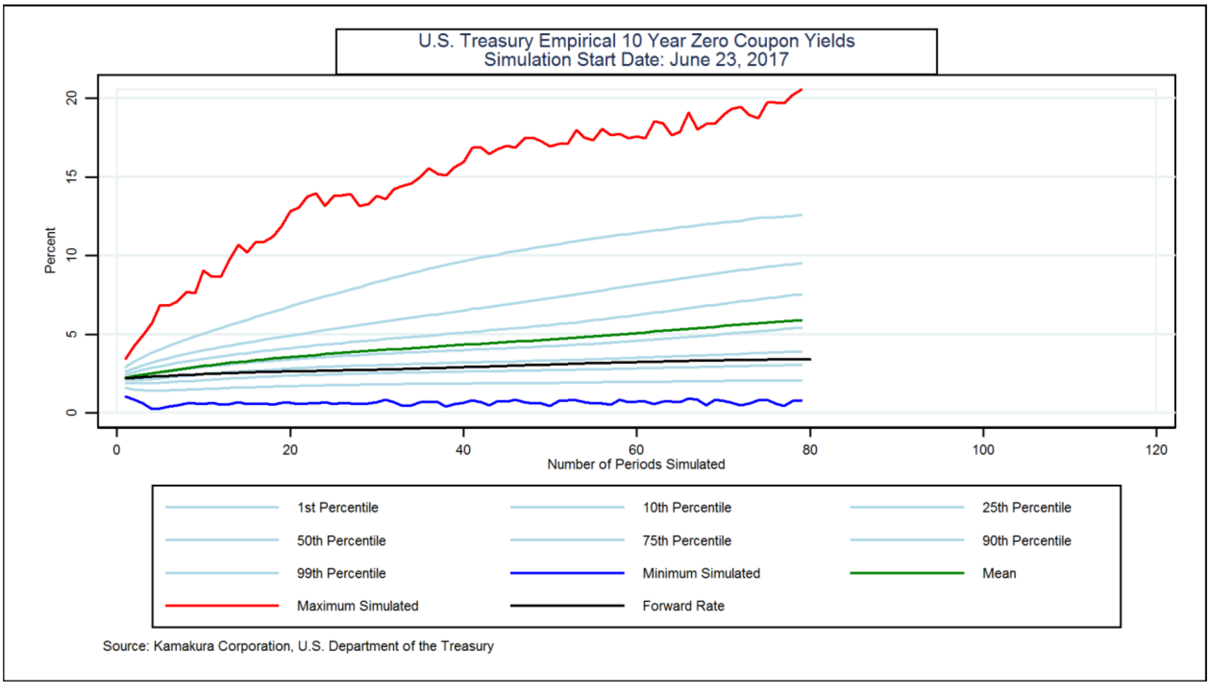
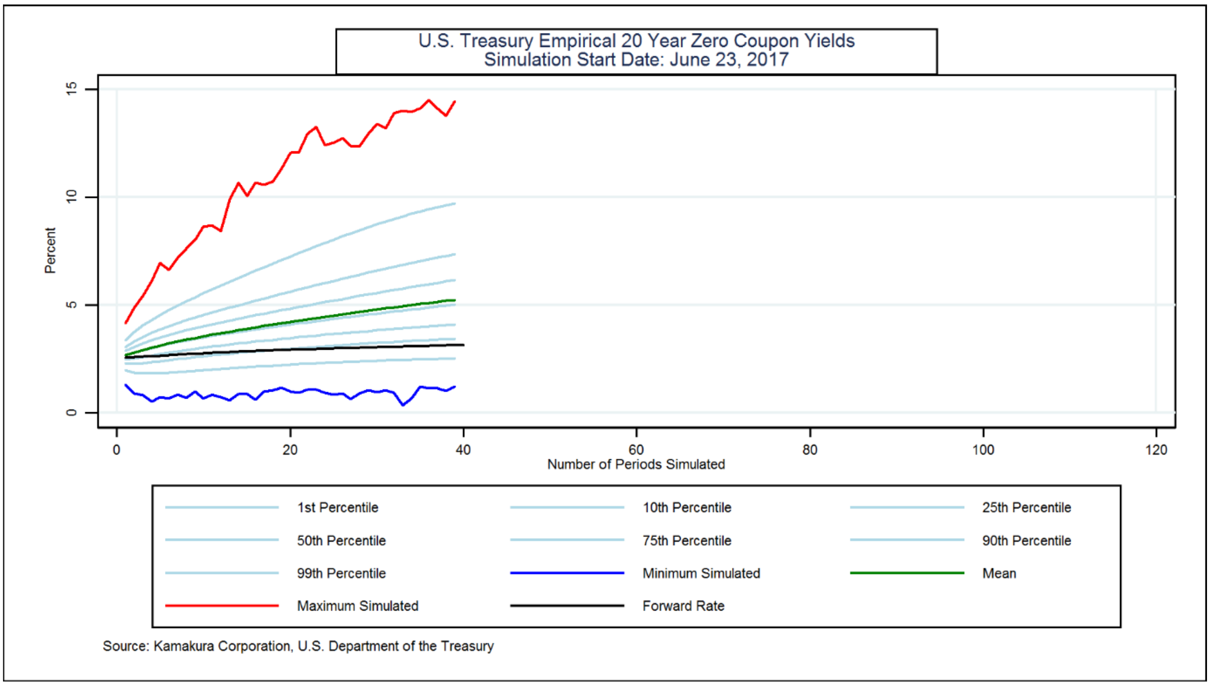


**Exhibit XXXI: 20 Year Yields**

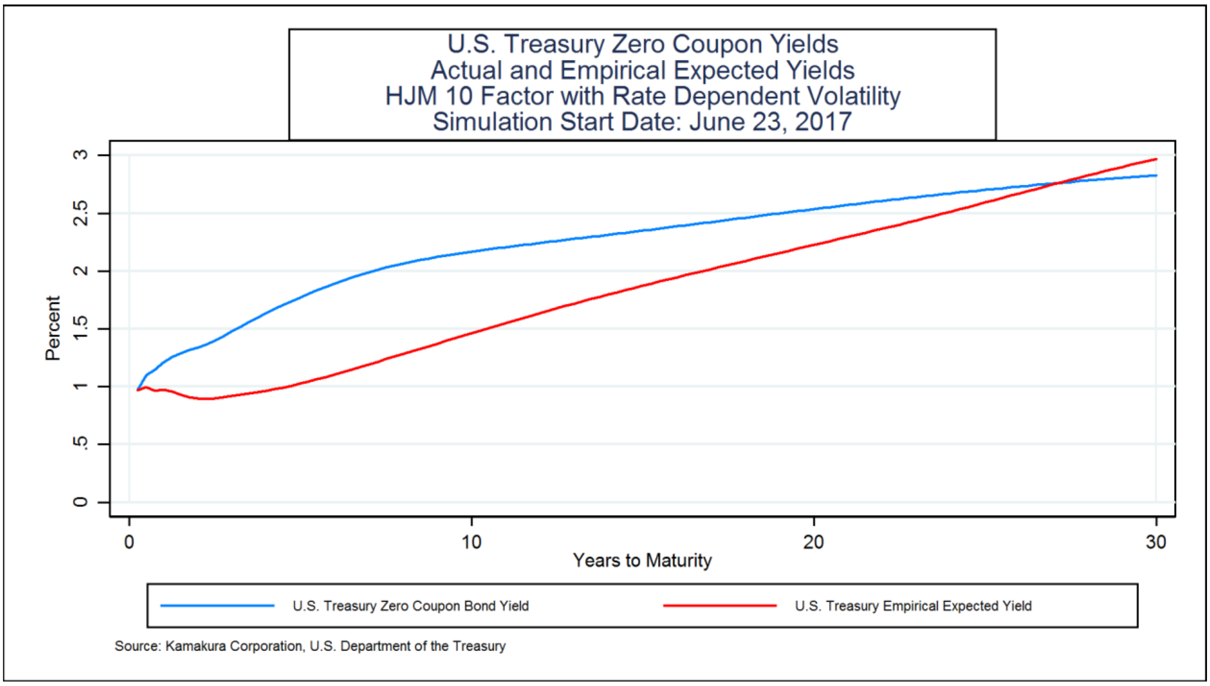



We again determine that there are no unexpected variations in the distribution of empirical yields over time.

## E. Simulation of the Term Premium

The size of the "term premium" of actual zero-coupon yields over the expected level of the short rate is a topic of great interest to both academics and policy makers. In a stochastic volatility model, the term premium must be determined by simulation because in general there is no closed form solution for expected future rates. The table below shows a term premium that widens initially, then narrows gradually as the simulation proceeds over time.

**Exhibit XXXII: Simulation of the Term Premium**



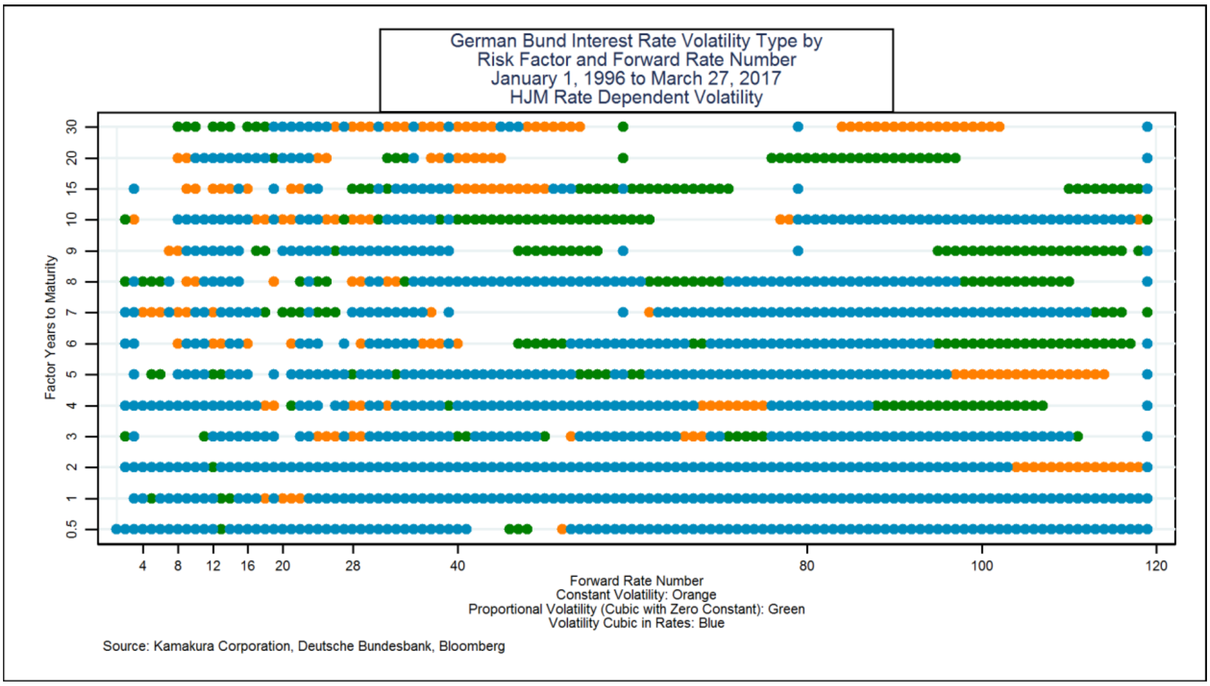

We again conclude that the simulation produces results that are consistent with the "scientific knowledge" about the variation in interest rates around the world.

## IV. Conclusion

The German Bund yield curve is driven by 14 factors, a number of factors very similar to government yield curves in ten other markets for which studies have been conducted. The January 1996 to March 2017 yield history for the Germany is both relatively long and spans a wide range of interest rate experience.

The stochastic volatility assumption provided more accurate and more reasonable parameters than a constant volatility model, particularly in the context of Bayesian simulations as part of the model validation process. Exhibit XXXIII summarizes the reasons for those conclusions:

**Exhibit XXXIII: Statistical Significance Summary and Volatility Classification**




The vertical axis lists the maturities used as risk factors by years to maturity of the underlying quarterly forward rate. The risk factors are the idiosyncratic movement of each of these forward rates. If the risk factor is statistically significant in explaining the movement of forward rates with the quarterly maturities listed on the horizontal axis, a dot is placed in the grid.

The nature of interest rate volatility for each combination of risk factor maturity and forward rate maturity is color coded. If the derived volatility is constant, the color code is orange. This is the affine specification. The graph shows immediately that a small minority of the risk factor maturity/forward rate maturity volatilities are consistent with the affine structure. The green and blue codes address the issue of whether interest rate volatility for that combination of risk factor maturity and forward rate maturity is zero or not when the forward rate is zero. If the measured volatility at a zero forward rate level is zero, the color code is green. Otherwise the color code is blue. In both cases, the volatility is a stochastic function of the forward rates at the start of the simulation period.

The chart summarizes the fact that all 14 factors are statistically significant across the yield curve for German Bund. The dominant derived interest rate volatility is the cubic stochastic volatility specification with a non-zero constant. An affine assumption for interest rate volatility is best fitting for a small minority of the combinations of risk factor maturity and forward rate maturity.

## Appendix

In spite of the overwhelming evidence across countries that government bond yields are driven by multiple factors, the use of single factor term structure models in interest rate risk management systems remains common even in some of the world's largest banks. This appendix asks and answers a number of important questions on the use of one factor models that any sophisticated model audit would pose. Given the answers below, most analysts would conclude that one factor term structure models are less accurate than a long list of multi-factor term structure models and that the one factor models would therefore fail a model audit.

We address two classes of one factor term structure models, all of which are special cases of the Heath, Jarrow and Morton framework, in this appendix using data from the German Bund market. Answers for other government bond markets cited in the references are nearly identical.

One factor models with rate-dependent interest rate volatility;
- Cox, Ingersoll and Ross (1985)
- Black, Derman and Toy (1990)
- Black and Karasinski (1991)

One factor models with constant interest rate volatility (affine models)
- Vasicek (1977)
- Ho and Lee (1986)
- Extended Vasicek or Hull and White Model (1990, 1993)

**Non-parametric test 1: Can interest rates be negative in the model?**

The one-factor models with rate-dependent interest rate volatility make it impossible for interest rates to be negative. Is this implication true or false? It is false, as Deutsche Bundesbank yield histories, Swedish Government Bond histories, Japanese Government Bond yields and yields in many other countries show frequent negative yields in in recent years. Table V and this video of forward rates and zero-coupon bond yields for the Japanese Government Bond yield curve documents the existence of negative forward rates using daily data from September 24, 1974 through December 30, 2016:

https://www.youtube.com/watch?v=X49I1rIZPJg

**Non-parametric test 2:** As commonly implemented, one-factor term structure models imply that all yields will either (a) rise, (b) fall, or (c) remain unchanged. This implication is false, as documented for the Germany in Table III. In fact, yield curves have twisted on 62% of the observations for the German Bund market.

**Non-parametric test 3**: The constant coefficient one-factor models imply that zero-coupon yields are normally distributed and so are the changes in zero-coupon yields. In the German Bund market, this implication is rejected by three common statistical tests for 120 of 120 quarterly maturities for zero yields and for 118 of the 120 of the quarterly forward return changes, as shown in Table II.

**Assertion A: There are no factors other than the short-term rate of interest that are statistically significant in explaining yield curve movements.** This assertion is false. Table VI shows, using principal components analysis, that 17-18 factors are needed to explain the movements of the German Bund Yield Curve. Exhibit IX makes the same point in more detail.

**Assertion B: There may be more than one factor, but the incremental explanatory power of the 2nd and other factors is so miniscule as to be useless.**

This assertion is false, as the 2nd through 14th factors in the German Bund market explain 42% of forward rate movements, compared to 58% for the first factor alone. In most countries, the best “first factor” is not the short rate of interest used by many large banks; it is the parallel shift factor of the Ho and Lee model.

**Assertion C: A one-factor “regime shift” model is all that is necessary to match the explanatory power of the 2nd and other factors.** This assertion is also false. A recent study prepared for a major U.S. bank regulator also confirmed that a one factor “regime shift” term structure model made essentially no incremental contribution toward resolving the persistent lack of accuracy in one factor term structure models.

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