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An 11 Factor Heath, Jarrow and Morton Model for the Thai Government Bond Yield Curve: Implications for Model Validation

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ABSTRACT

This paper analyzes the number and the nature of factors driving the movements in the Thai Government Bond yield curve from September 15, 1999 through December 31, 2016. The process of model implementation reveals a number of important insights for interest rate modeling generally. First, model validation of observed yields is important because those yields are the product of a third-party curve fitting process that may produce spurious measures of interest rate volatility. Second, quantitative measures of smoothness and international comparisons of smoothness provide a basis for measuring data quality. Third, we outline a process for re-smoothing the raw data in a manner that preserves the maximum amount of true signal within that data. Finally, we illustrate the process for comparing stochastic volatility and affine models of the term structure. We conclude that the relatively short history of the data series in Thailand and the relatively narrow range of rate variation implies a constant volatility or "affine" specification, unlike other markets where stochastic volatility models have a superior fit.

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An 11 Factor Heath, Jarrow and Morton Model for the Thai Government Bond Yield Curve: Implications for Model Validation

Government yield curves are a critical input to the risk management calculations of major banks, insurance firms, fund managers, pension funds, and endowments around the world. With the internationalization of fixed income investing, it is important to understand the dynamics of movements in yield curves outside of the major bond markets like those in Frankfurt, London, New York and Tokyo. In this paper, we fit a multi-factor Heath, Jarrow and Morton model to daily data from the Thai Government Bond market over the period from September 15, 1999 to December 31, 2016. The modeling process reveals a number of important implications for term structure modeling in other government bond markets.

Section I discusses the origin and characteristics of the daily data base of Thai Government Bond yields provided by the Thai Bond Market Association. Model validation on the raw data in the data base reveals a higher degree of variation in forward rates, even when fit on a "maximum smoothness" basis, than is typical of international markets. We quantify the differences in smoothness by defining a discrete model-independent measure of smoothness and comparing this measure for the Thai and U.S. Treasury yield curves. We conclude that the underlying Thai data includes spurious variation in forward rates due to the original yield curve smoothing methodology. We then limit the yield curves used as inputs to the secondary smoothing process to maturities that maximize the use of "on the run" bond yields, yields of the most recently issued Thai Government Bonds. We present videos comparing the original and revised data and present a comparison, also in video form, of the original Thai forward rates and U.S. Treasury forward rates. We also compare the smoothness measures of Thai and U.S. Treasury yield curves. We conclude that the revised Thai data provides the best basis for fitting a multi-factor Heath, Jarrow and Morton model.

Section II outlines the process for determining whether the interest rate volatility for the factors driving the Thai yield curve is constant (an "affine" model) or stochastic, typically expressed as a function of the level of interest rates. We conclude that the Thai market, unlike other markets studied, is consistent with the affine specification. Section III describes the process of fitting five different Heath, Jarrow and Morton models to Thai Government bond yield data: models with 1, 2, 3, 6 and 11 factors. Section IV concludes the paper. The Appendix illustrates a sample model validation process for widely used one factor term structure models using Thai, Japanese and U.S. data.

I. Thai Government Bond Data: Special Characteristics

A multi-factor term structure model is the foundation for best practice asset and liability management, market risk, economic capital, interest rate risk in the banking book, stress-testing and the internal capital adequacy assessment process. The objective in this paper is to illustrate the derivation of a multi-factor Heath Jarrow and Morton model of the Thai Government Bond yield curve. As a by-product, the analysis reveals common data problems associated with yield curve histories and requires a standard methodology for quantification and resolution of those problems. Previous implementations of multi-factor Heath, Jarrow and Morton models have covered the following bond market sectors:

Australia	Commonwealth Government Securities
Canada	Government of Canada Securities
Germany	German Bunds
Japan	Japanese Government Bonds
Singapore	Singapore Government Securities
Spain	Spanish Government Bonds
Sweden	Swedish Government Securities
United Kingdom	United Kingdom Government Bonds
United States	U.S. Treasury Securities

The first step in data model validation for the Thai Government Bond market is to examine the historical availability of bond yields over time. This availability is summarize in Table I.

Table I

Thai Government Bond Yield Analysis
Data Regime History
Source: Kamakura Corporation, Thai Bond Market Association

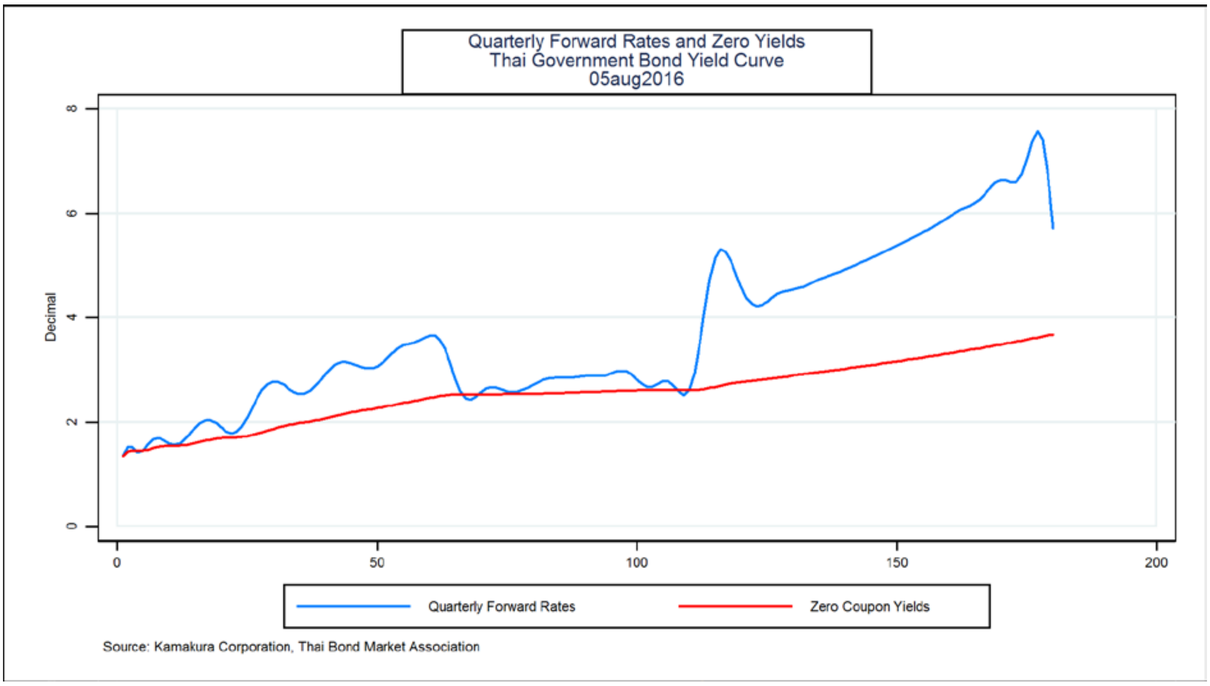
Data Regime Range		Maturities Available from Thai Bond Market Association				Regime
First Date	End Date	1 Month	3 Months	6 Months	Annual from	Number
9/15/1999	3/8/2000				1 year to 14 years	1
3/9/2000	7/5/2000				1 year to 13 years	1
7/6/2000	2/28/2001				1 year to 14 years	1
3/1/2001	4/19/2001	Yes	Yes		1 year to 14 years	1
4/20/2001	4/12/2002	Yes	Yes		1 year to 19 years	2
4/17/2002	6/28/2002	Yes	Yes		1 year to 18 years	2
7/2/2002	11/7/2002	Yes	Yes	Yes	1 year to 18 years	2
11/8/2002	11/13/2003	Yes	Yes	Yes	1 year to 19 years	2
11/14/2003	11/12/2004	Yes	Yes	Yes	1 year to 18 years	2
11/15/2004	11/11/2005	Yes	Yes	Yes	1 year to 17 years	2
11/14/2005	7/6/2006	Yes	Yes	Yes	1 year to 16 years	2
7/7/2006	7/12/2007	Yes	Yes	Yes	1 year to 19 years	2
7/13/2006	11/29/2007	Yes	Yes	Yes	1 year to 18 years	2
11/30/2007	3/18/2008	Yes	Yes	Yes	1 year to 20 years	2
3/19/2008	4/24/2008	Yes	Yes	Yes	1 year to 19 years	2
4/25/2008	3/20/2009	Yes	Yes	Yes	1 year to 29 years	3
3/23/2009	6/23/2009	Yes	Yes	Yes	1 year to 28 years	3
6/24/2009	10/13/2009	Yes	Yes	Yes	1 year to 29 years	3
10/14/2009	6/30/2010	Yes	Yes	Yes	1 year to 30 years	3
7/2/2010	10/26/2010	Yes	Yes	Yes	1 year to 29 years	3
10/27/2010	3/1/2011	Yes	Yes	Yes	1 year to 30 years	3
3/2/2011	6/30/2011	Yes	Yes	Yes	1 year to 50 years	4
7/4/2011	6/29/2012	Yes	Yes	Yes	1 year to 49 years	4
7/2/2012	6/28/2013	Yes	Yes	Yes	1 year to 48 years	4
7/2/2013	6/27/2014	Yes	Yes	Yes	1 year to 47 years	4
6/30/2014	6/29/2015	Yes	Yes	Yes	1 year to 46 years	4
6/30/2015	11/24/2015	Yes	Yes	Yes	1 year to 45 years	4
11/25/2015	6/29/2016	Yes	Yes	Yes	1 year to 50 years	4
6/30/2016		Yes	Yes	Yes	1 year to 49 years	4

The data shows that the Thai Bond Market Association's data history is unusual in its frequent changes in "data regime," i.e. which the maturities are available on a given date. The data is also unusual in the granularity of the yield curve data. By June 30, 2016, the Thai Bond Market Association was providing 3 short term yields and 49 yields of 1 year or more, the highest degree of granularity of the government bond markets studied so far.

Because the Heath, Jarrow and Morton analysis makes use of a yield curve with quarterly forward rate segments, the next step in data model validation is to fit quarterly forward rates to the raw coupon-bearing bond yields. The smoothness of the resulting forward rates will be a function of both the quality of the raw data from a smoothness point of view and the smoothness implied by the secondary smoothing process. To ensure the maximum smoothness from the secondary smoothing process, we use the maximum smoothness forward rate methodology of Adams and van Deventer [1994], as corrected in van Deventer and Imai [1996]. Adams and van Deventer show that the maximum smoothness method overcomes the problems of the cubic spline approach of McCulloch, and, unlike the Svensson [1994] approach, allows for a perfect fit to the raw data provided by the Thai Bond Market Association. See Jarrow [2014] for information on the problems with Svensson yield curve fitting.

We then conduct a visual inspection of the resulting forward rates implied by the raw data. The yield curve on August 5, 2016 in Exhibit I is representative:

Exhibit I



The forward rate curve are the smoothest curve that can be fit to the raw data provided by the Thai Bond Market Association, but it implies much more forward rate variation than is typical for government yield curves. A video of the quarterly forward rates (in blue) versus the zero coupon bond yields implied by the Thai Bond Market Association on every business day is given here:

https://www.youtube.com/watch?v=yfzeu7Mts2o

If we count the daily local maxima and minima in the Thai Government Bond forward rate curve implied by the data, we get many more "humps" than is typical in other markets. We can make this examination qualitatively by comparing the shape of the implied quarterly forward rates in the Thai Government Bond market and the U.S. Treasury market on the same day from September 15, 1999 through December 31, 2016, as in this video:

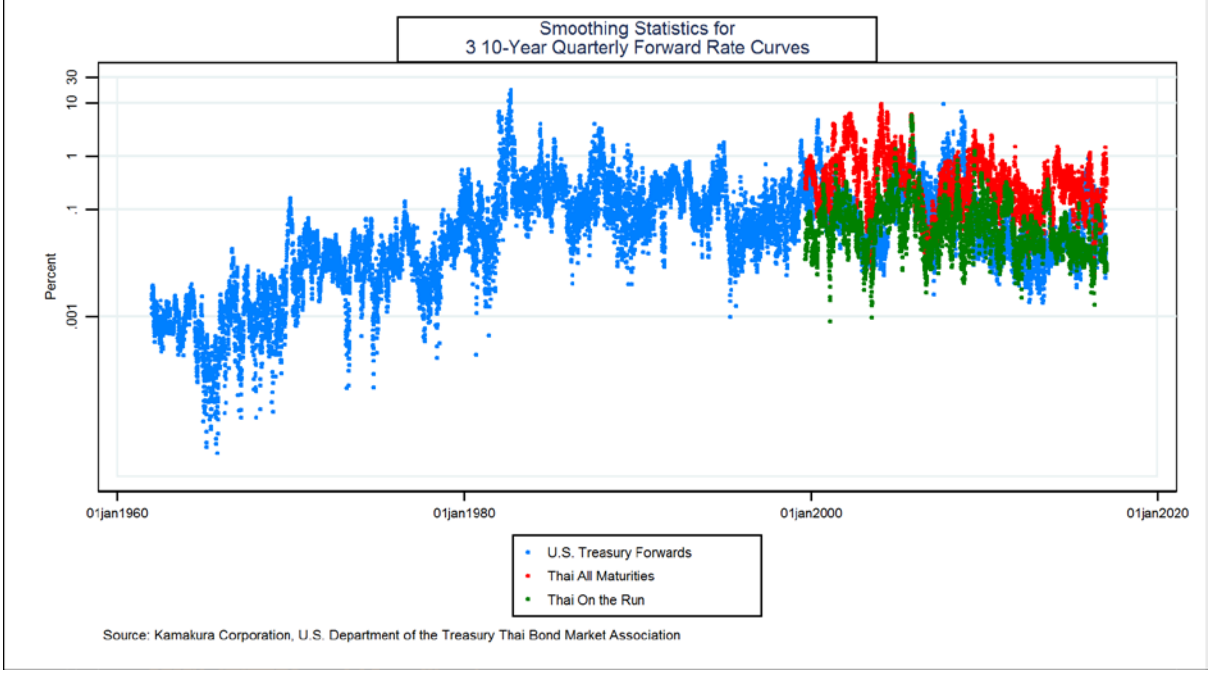
https://youtu.be/MtHEC25Xa50

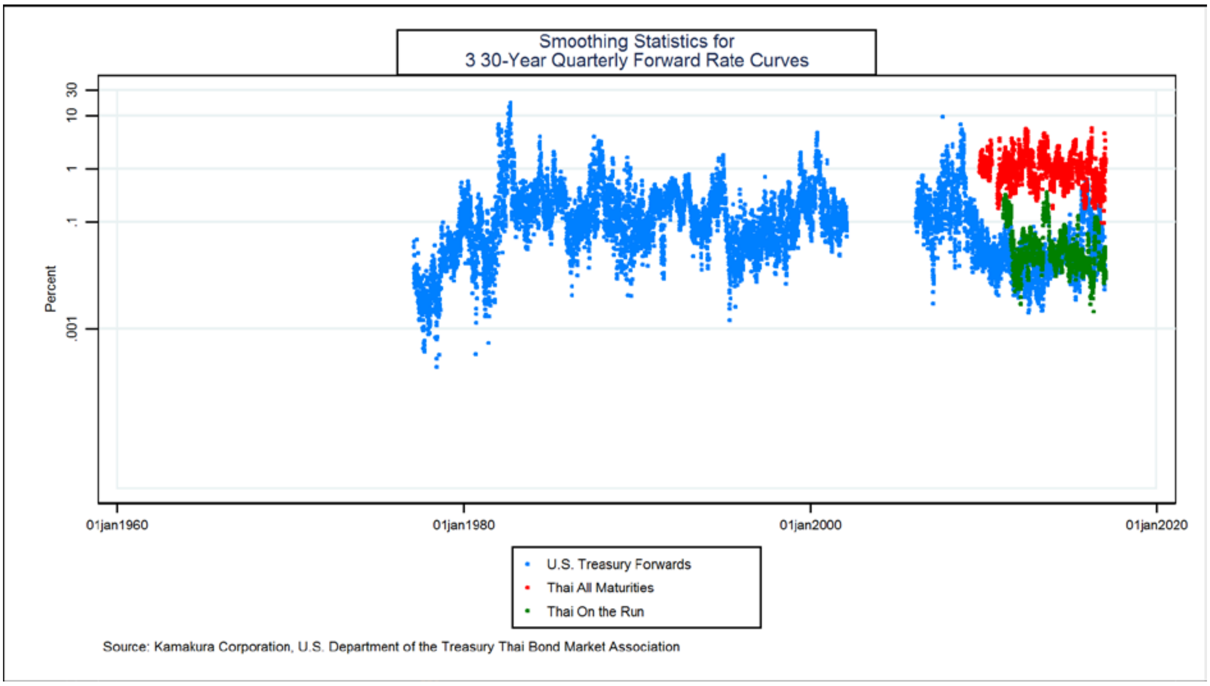
Qualitatively, the Thai Government forward rate curve is much more volatile than U.S. Treasury forward rates. The smoothness of the quarterly forward rate curve can be measured quantitatively using the quarterly forward rates implied by the Thai Bond Market and U.S. Treasury yield curves. For a yield curve that consists of N quarterly forward rates, the discrete smoothness statistic at time t $Z_N(t)$ is the sum of the squared second differences in the forward rates, as explained by Adams and van Deventer [1994]. A continuous smoothing statistic can also be calculated when the functional form of the continuous forward rate is known.

$$Z_N(t) = \sum_{i=3}^{N} \left[\left(f_i(t) - f_{i-1}(t)\right) - \left(f_{i-1}(t) - f_{i-2}(t)\right) \right]^2$$

We calculated this smoothness statistic for the first 10 years and the first 30 years of both the Thai Government Bond yield curve and the U.S. Treasury yield curve. The results are given in Exhibit II on a log scale:

Exhibit II





We conclude that the raw data provided by the Thai Bond Market Association implies unrealistic movements in forward rates. We seek to preserve the key insights of the data while removing the spurious volatility it implies. We do that by using only those long term maturities at which the Thai Government actually issues securities: 5 years, 10 years, 20 years, 30 years, and 50 years. We adjust these maturities for the maximum availability of data as the bonds' maturities shorten. That modified list of "almost on the run" maturities includes data for 1 month, 3 months, 6 months, 1 year, 1.5 years, 2 years, 3 years, 5 years, 10 years, 16 years, 28 years, and 45 years. These abridged maturities were used for a modified smoothing process. The smoothness of the revised forward rates (in blue) is compared with the smoothness of the original forward rates (in red) in this video:

https://youtu.be/FZZYVjeXxco

Exhibit II shows that the 10 year and 30 year smoothness statistics (in green) for the revised Thai forward rates is comparable to U.S. Treasuries:

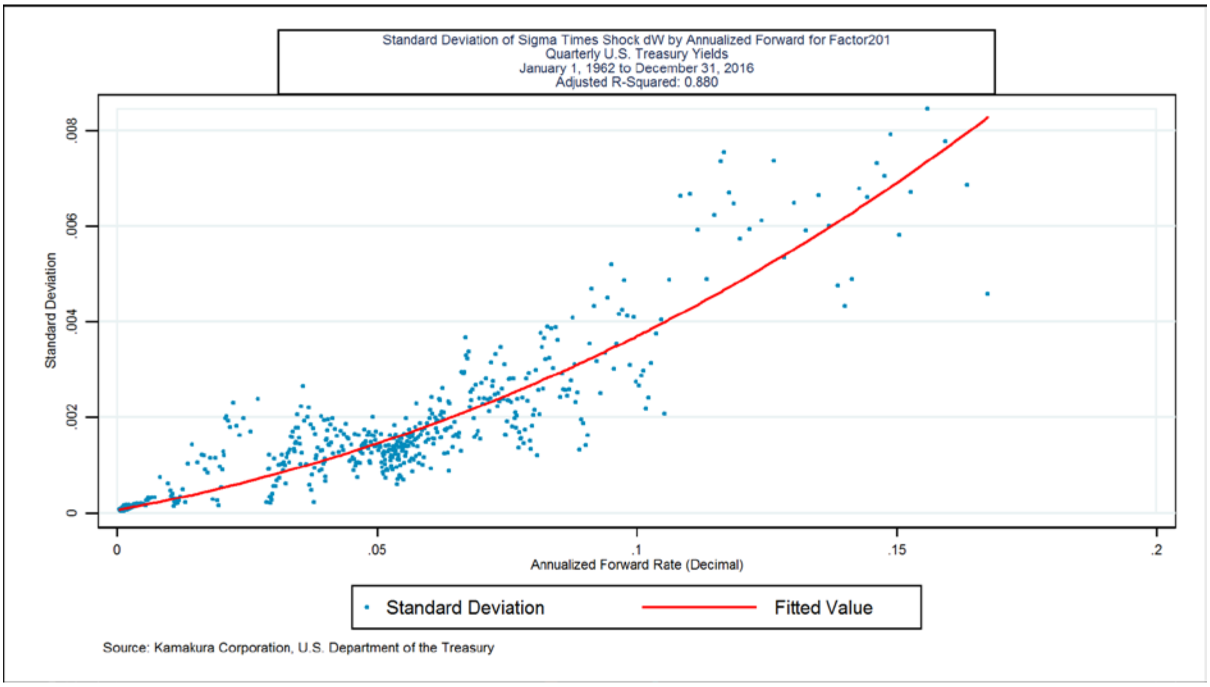
Finally, the actual "on the run" quarterly forward rates and zero coupon bond yields used to fit the Heath, Jarrow and Morton models are given in this video on a daily basis from September 15, 1999 to December 31, 2016:

https://youtu.be/6ExSbgfnnUw

II. Constant versus Stochastic Volatility

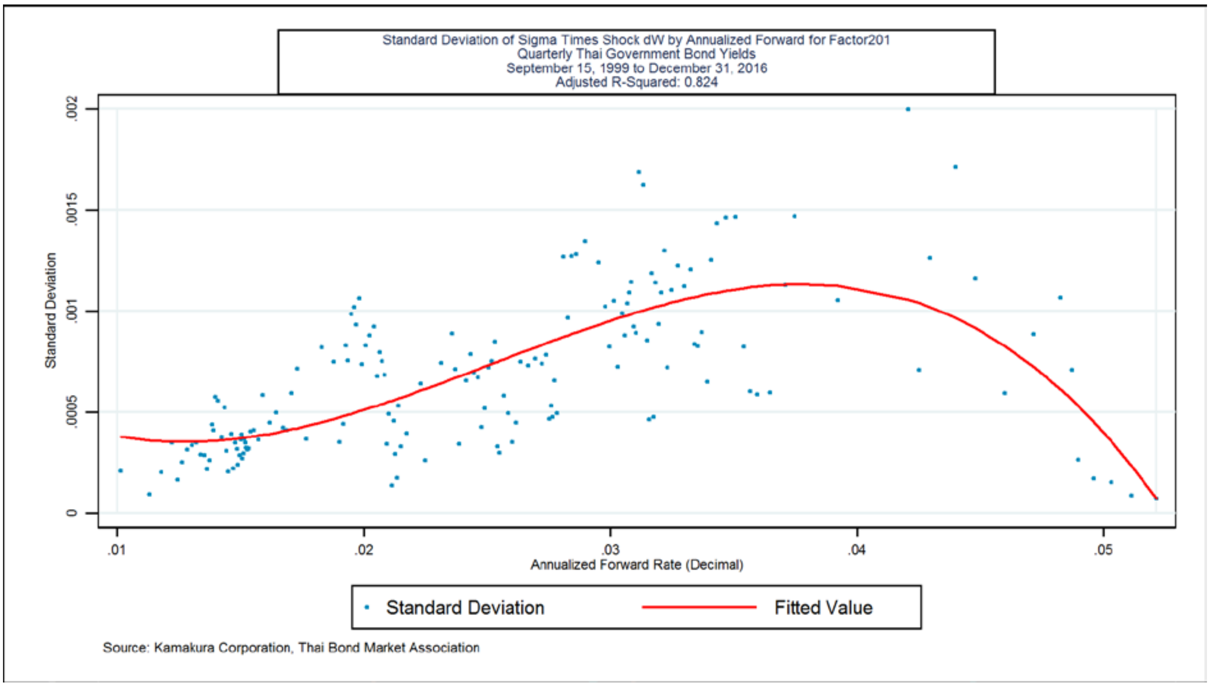
Constant volatility ("affine") term structure models are commonly used for their ease of simulation and estimation of "future expected rates" in order to determine the "term premium" in current yields. Prominent examples are Adrian, Crump and Moench [2013], Kim and Wright [2005], and Duffie and Kan [1996]. On the other hand, the weight of the empirical evidence in most of the countries studied to date indicates that interest rate volatility does vary by the level of the corresponding forward rate. To illustrate that fact, we studied the shortest forward rate on the U.S. Treasury curve on a daily basis from January 2, 1962 through December 31, 2016. We ordered the data from lowest forward rate level to highest forward rate level. We formed non-overlapping groups of 25 observations each and calculated both the standard deviation of 91 day forward rate changes and the mean beginning-of-period forward rate in each group. The results are plotted in Exhibit III:

Exhibit III



A cubic function of annualized forward rates explains 88% of the variation in the standard deviation of forward rate changes for these ordered groups. This phenomenon has been confirmed in the government securities markets for Australia, Canada, Germany, Japan, Singapore, Spain, Sweden, the United Kingdom, and the United States. Exhibit IV shows the results for Thailand:

Exhibit IV



While each of the coefficients of the cubic function of annualized forward rates is statistically significant, the implied shape of the stochastic volatility function is unreasonable. We reject it as an unacceptable forecast for future interest rate volatility under standard Bayesian application of "scientific knowledge" to out of sample data.[2] We judge a constant volatility model to be more accurate and focus on affine versions of the Heath, Jarrow and Morton model in what follows.

Using the on the run maturities for Thai Government Bond yields, the maximum smoothness forward rate approach generates this daily evolution of Thai Government Bond zero coupon yields over time:

[2] See Gelman et al, page 3, for an introduction to the 3 principal steps in Bayesian analysis.

Exhibit V

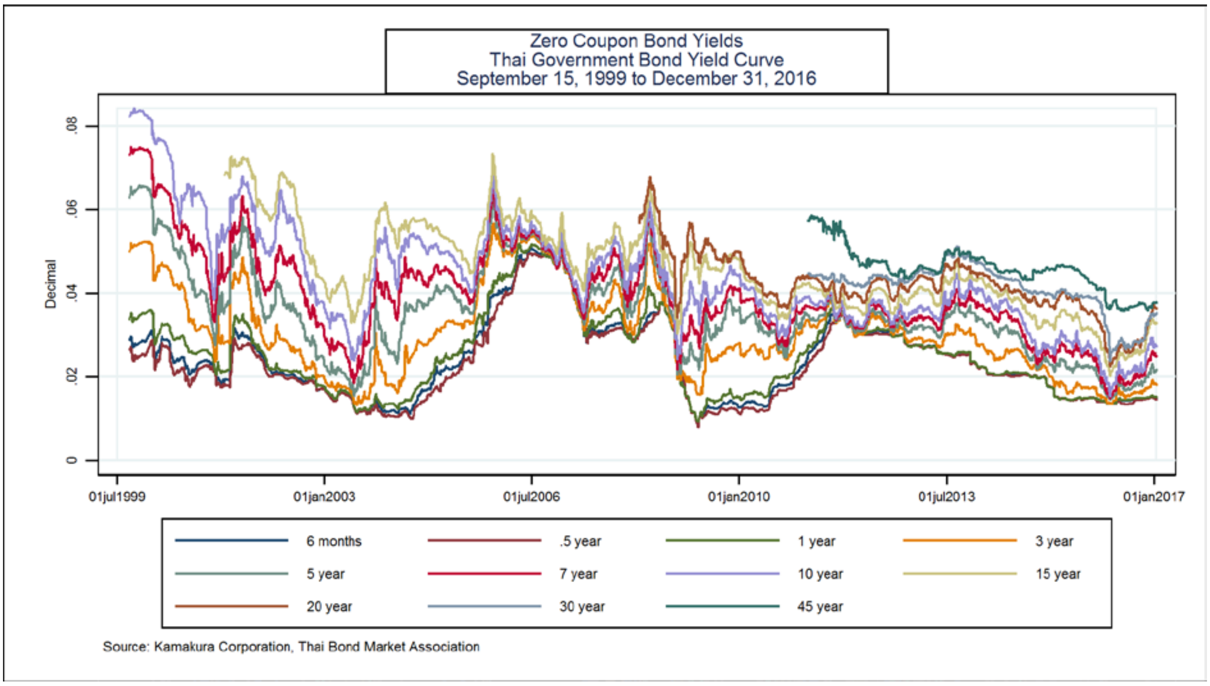
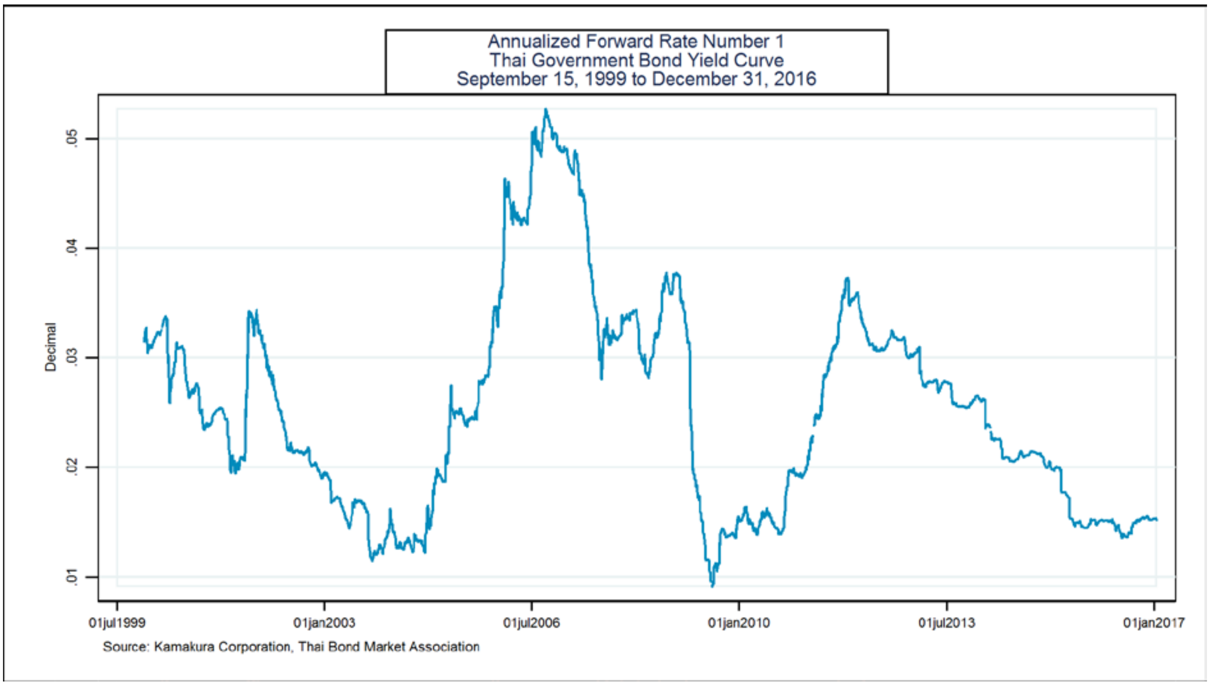


Exhibit VI below shows the evolution of the first quarterly forward rate (the forward that applies from the 91st day through the 182nd day) over the same time period:

Exhibit VI



We use three statistical tests to determine whether or not the hypothesis of normality should be rejected at the 5% level for two sets of data: the [Shapiro-Wilk](#) test, the [Shapiro-](#)

Francia test, and the skew test, all of which are available in common statistical packages. The results of these tests are summarized in Table II:

Table II

HJM 11 Factor Model
Thai Government Bond
Using Daily Data from September 15, 1999 through December 31, 2016
Date of Analysis: February 9, 2017

NORMALITY OF ZERO COUPON BOND YIELD ABSOLUTE LEVELS AND CONTINUOUSLY COMPOUNDED CHANGES IN FORWARD RETURNS

P-values for Null Hypothesis that Zero Coupon Bond Yields are Normally Distributed				P-values for Null Hypothesis that Discrete Changes in Forward Returns Are Normally Distributed				Z Yield Result	Return Result	Number Rejected: Z Yield Result	Number Rejected: Return Result
Quarter	Shapiro Wilk Test	Shapiro Francia Test	Skew Test	Quarter	Shapiro Wilk Test	Shapiro Francia Test	Skew Test	Is Normality Hypothesis Rejected?	Is Normality Hypothesis Rejected?	Is Normality Hypothesis Rejected?	Is Normality Hypothesis Rejected?
1	0.000%	0.000%	0.000%	1	0.000%	0.000%	0	Yes	Yes	172	179
2	0.000%	0.000%	0.000%	2	0.000%	0.000%	0	Yes	Yes		
3	0.000%	0.000%	0.000%	3	0.000%	0.000%	0	Yes	Yes		
4	0.000%	0.000%	0.000%	4	0.000%	0.000%	0	Yes	Yes		
5	0.000%	0.000%	0.000%	5	0.000%	0.000%	0	Yes	Yes		
6	0.000%	0.000%	0.000%	6	0.000%	0.000%	0.000%	Yes	Yes		
7	0.000%	0.000%	0.000%	7	0.000%	0.000%	0.000%	Yes	Yes		
8	0.000%	0.000%	0.000%	8	0.000%	0.000%	0.000%	Yes	Yes		
9	0.000%	0.000%	0.000%	9	0.000%	0.000%	0.000%	Yes	Yes		
10	0.000%	0.000%	0.000%	10	0.000%	0.000%	0.000%	Yes	Yes		
11	0.000%	0.000%	0.000%	11	0.000%	0.000%	0.000%	Yes	Yes		
12	0.000%	0.000%	0.000%	12	0.000%	0.000%	0.000%	Yes	Yes		

Table II above shows the p-values for these three statistical tests for the first twelve quarterly maturities. We conduct the test for each quarter out to 45 years, the longest maturity used in the smoothing process. The null hypothesis of normality is rejected by all 3 tests for 172 of the 180 quarterly zero coupon yield maturities. For quarterly changes in forward rates, the null hypothesis of normality is again rejected by all 3 tests for all 179 of the 179 maturities for changes in forward rates. This is a powerful rejection of the normality assumptions implicit in constant coefficient or "affine" term structure models. In most of the other countries studied, the hypothesis of normality has been rejected strongly as well. Given these results, we proceed with caution on the implementation of the affine model.

In Chapter 3 of Advanced Financial Management (second edition, 2013), van Deventer, Imai and Mesler analyze the frequency with which U.S. Treasury forward rates move up together, down together or remain unchanged. This exercise informs the Heath, Jarrow and Morton parameter fitting process and is helpful for the model validation questions posed in the Appendix. We perform the yield curve shift analysis using 4,236 days of zero coupon bond yields for the Thai Government Bond yield curve. We analyze the daily shifts in the zero coupon bond yields on each business day from September 15, 1999 through December 31, 2016. The results are given in Table III:

Table III

HJM 11 Factor Model Thai Government Bond Using Daily Data from September 15, 1999 through December 31, 2016 Date of Analysis: February 9, 2017		
Type of Yield Shift	**Number of Observations**	**Percent of Observations**
All yields shift up	252	5.95
All yields shift down	42	0.99
All yields are unchanged	0	0.00
Yield curve twists	3,942	93.06
Total	4,236	100.00

Kamakura Corporation, Thai Bond Market Association

Yield curve shifts were all positive, all negative, or all zero 5.95%, 0.99%, and 0% of the time, a total of 6.94% of all business days. The predominant yield curve shift was a twist, with a mix of positive changes, negative changes, or zero changes. These figures are similar to those for the U.S. Treasury, Japanese Government Bond, Government of Canada, and United Kingdom Government Bond yield curves. These twists, which happen 93.06% of the time in Thailand, cannot be modeled at all with the conventional implementation of one factor term structure models.

Another important aspect of yield curves is the number of local minima and maxima that have occurred over the modeling period. The results for the Thailand Government Bond Market are given in Table IV:

Table IV

Analysis of Number of Local Minima and Maxima Each Day		
Number of Humps	**Number of Observations**	**Percent of Observations**
0 local minimum and maximum	2,198	51.89
1	447	10.55
2	994	23.47
3	337	7.96
4	155	3.66
5	61	1.44
6	26	0.61
7	16	0.38
8	2	0.05
9	0	0.00
10 or more	0	0.00

Kamakura Corporation, Thai Bond Market Association

The number of days with 0 or 1 humps (defined as the sum of local minima and maxima on that day's yield curve) was 62.44% of the total observations in the data set.

Finally, before proceeding, we count the number of occurrences of negative rates for each forward rate segment of the yield curve over the history provided by the Thai Bond Market Association. As of December 31, 2016, there are no observations of negative forward rates in the Thai Government Bond market.

III. Fitting Heath, Jarrow and Morton Parameters

A simple first step in constructing a multi-factor Heath, Jarrow and Morton model is to conduct principal components analysis on the forward rates that make up the relevant yield curve. For the Thai Government Bond market, at its longest maturity, these quarterly segments consist of one three-month spot rate and 199 forward rates. Over 1363 observations, the principal components analysis indicates in Table II that the first factor explains only 36.77% of the movement in forward rates over the full curve. For a high degree of explanatory power, the principal components analysis indicates that 10 to 12 factors will be necessary.

Table V

Principal components/correlation

Rotation: (unrotated = principal)

Number of obs	= 1,363
Number of comp.	= 16
Trace	= 179
Rho	= 1.0000

Component	Eigenvalue	Difference	Proportion	Cumulative
Comp1	65.8216	25.5815	0.3677	0.3677
Comp2	40.2401	3.57086	0.2248	0.5925
Comp3	36.6692	15.8769	0.2049	0.7974
Comp4	20.7923	13.4675	0.1162	0.9135
Comp5	7.32479	3.3376	0.0409	0.9545
Comp6	3.98719	2.36924	0.0223	0.9767
Comp7	1.61795	.371796	0.0090	0.9858
Comp8	1.24615	.355136	0.0070	0.9927
Comp9	.891017	.621565	0.0050	0.9977
Comp10	.269452	.166686	0.0015	0.9992
Comp11	.102766	.0775689	0.0006	0.9998
Comp12	.025197	.0158601	0.0001	0.9999
Comp13	.00933691	.00743574	0.0001	1.0000
Comp14	.00190117	.00104433	0.0000	1.0000
Comp15	.000856837	.00068637	0.0000	1.0000
Comp16	.000170466	.000162851	0.0000	1.0000
Comp17	7.61471e-06	5.38286e-06	0.0000	1.0000
Comp18	2.23185e-06	5.83727e-07	0.0000	1.0000
Comp19	1.64813e-06	1.64813e-06	0.0000	1.0000
Comp20	0	0	0.0000	1.0000

With this analysis as background, we begin the Heath, Jarrow and Morton fitting process.

In the studies done so far, the number of statistically significant factors are summarized below:

Australia:	Commonwealth Government Securities,	14 factors
Canada:	Government of Canada Securities,	12 factors
Germany:	Bunds,	14 factors
Japan:	Japanese Government Bonds,	16 factors
Singapore:	Singapore Government Securities	9 factors
Spain	Spanish Government Securities	11 factors
Sweden:	Swedish Government Securities,	11 factors
United Kingdom:	Government Securities,	14 factors
United States:	Treasury Securities,	10 factors

We now fit a multi-factor Heath, Jarrow and Morton model to Thai Government Bond zero coupon yield data from September 15, 1999 to December 31, 2016. We use the four data regimes numbered in the right hand column of Table I. The first is for observations where no maturity longer than 14 years was reported. The second is for those observations where no maturity longer than 20 years was reported. The third regime includes observations with maturities as long as 30 years. The fourth regime includes maturities out to 50 years. The availability of data out to 50 years is fairly unusual in government bond markets world-wide.

The procedures used to derive the parameters of a Heath, Jarrow and Morton model are described in detail in Jarrow and van Deventer (June 16, 2015, November 1, 2016 and February 10, 2017).

We followed these steps to estimate the parameters of the model:

- We extract the zero coupon yields and zero coupon bond prices for all quarterly maturities out to 45 years for all daily observations for which the 45 year zero coupon yield is available. For other observations, we extended the analysis to the longest maturity available, which varies by data regime. This is done using Kamakura Risk Manager, version 8.1, using the maximum smoothness forward rate approach to fill the quarterly maturity gaps in the zero coupon bond data.
- We use overlapping 91-day intervals to measure changes in forward rates, avoiding the use of “quarterly” data because of the unequal lengths of calendar quarters. Because overlapping observations trigger auto-correlation, “HAC” (heteroscedasticity and autocorrelation consistent) standard errors are used. The methodology is that of Newey-West with 91 day lags.
- We consider eleven potential explanatory factors: the idiosyncratic portion of the movements in quarterly forward rates that mature in 6 months, 1 year, 1.5 years, 2, 3, 5, 7, 10, 16, 28 and 45 years. The Thai Government’s long bond issuance maturities are 20, 30 and 50 years, but those issues are infrequent. To maximize data availability, we use the slightly shorter maturities. Eleven factors is one more factor than the number of factors required by the Bank for International Settlements market risk guidelines published in January 2016.
- We calculate the discrete changes in forward returns as described in the parameter technical guide. Because the discrete changes are non-linear in the no-arbitrage framework of Heath, Jarrow and Morton, we use non-linear least squares to fit interest rate volatility.
- We use a different non-linear regression for each segment of the yield curve. We considered a panel-based approach, but we rejected it for two reasons: first, the movement of parameters as maturity lengthens is complex and not easily predictable before estimation; second, the residual unexplained error in forward rates is very, very small, so the incremental merit of the panel approach is minimal.
- We then begin the process of creating the orthogonalized risk factors that drive interest rates using the Gram-Schmidt procedure. These factors are assumed to be uncorrelated independent random variables that have a normal distribution with mean zero and standard deviation of 1.
- Because interest volatility is assumed to be stochastic, simulated out-of-sample forward rates will not in general be normally distributed. We also calculate constant volatility parameters and choose the most accurate from the constant volatility and stochastic volatility models estimated.
- In the estimation process, we added factors to the model as long as each new factor provided incremental explanatory power. The standard suite of models in both cases includes 1 factor, 2 factors, 3 factors, 6 factors and “all factors,” which varies by country.

We postulate that interest rate volatility for each forward rate maturity k is a cubic function of the annualized forward rate that prevails for the relevant risk factor j at the beginning of each 91-day period:

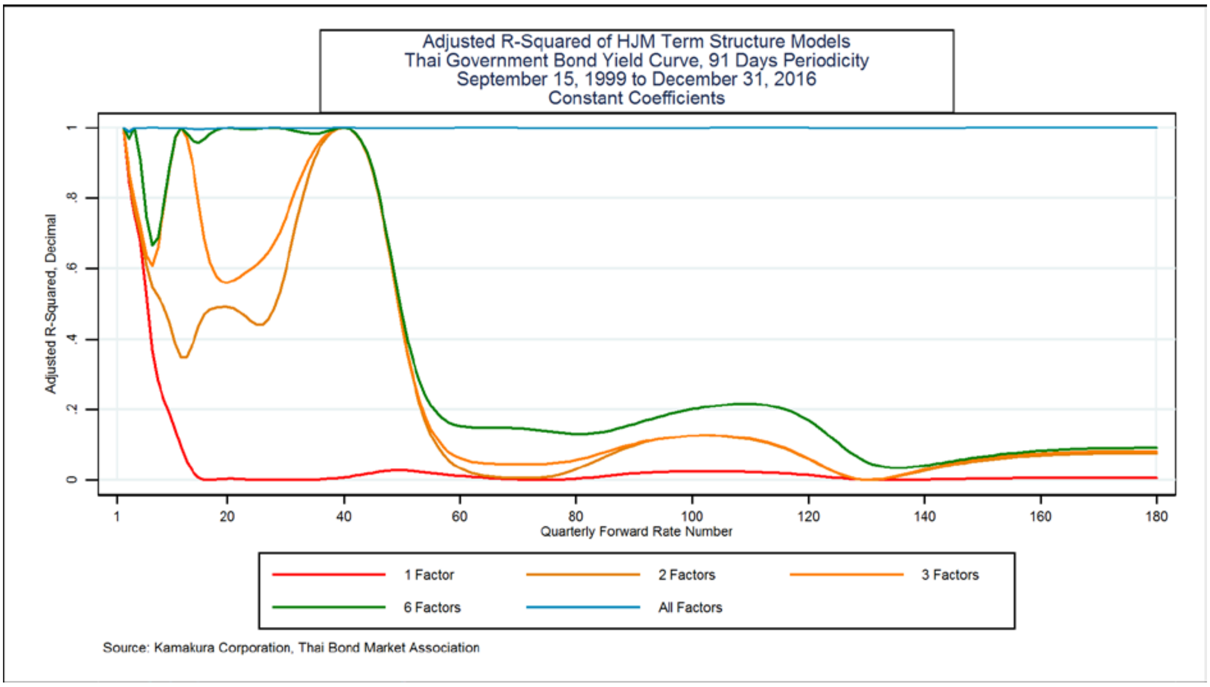
$$\sigma_{jk} = b_{0,jk} + b_{1,jk} f + b_{2,jk} f^2 + b_{3,jk} f^3$$

We use the resulting parameters and accuracy tests to address the hypothesis that a one factor model is "good enough" for modeling Thai Government Bond yields in the Appendix. We report the results for 1, 2, 3, 6 and 11 factors. The factors are the idiosyncratic variation in quarterly forward rates at each of 11 maturities. The factors, described by the maturity of the forward rate used, are added to the model in this order:

Factor 1: 6 months
Factor 2: 10 years
Factor 3: 3 years
Factor 4: 7 years
Factor 5: 1 year
Factor 6: 5 years
Factor 7: 2 years
Factor 8: 1.5 years
Factor 9: 16 years
Factor 10: 28 years
Factor 11: 46 years

Exhibit VII summarizes the adjusted r-squared for the non-linear equations for each of the 179 quarterly forward rate segments that make up the Thai Government Bond yield curve:

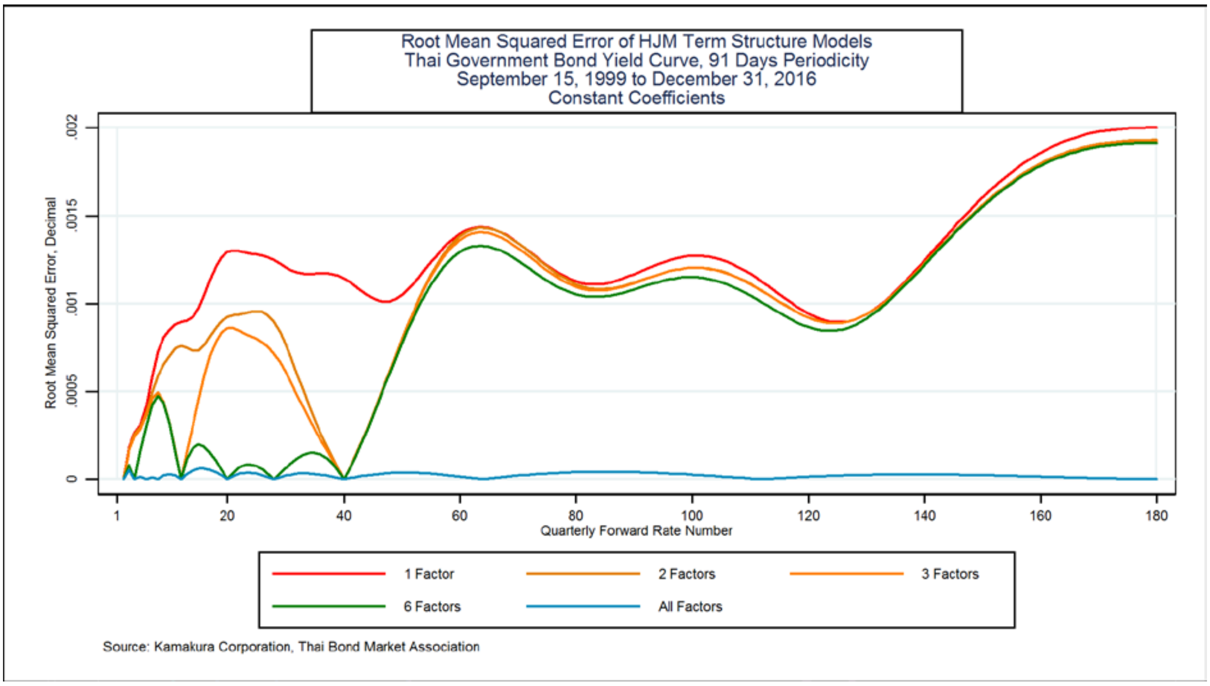
Exhibit VII



The adjusted r-squared for the best practice model over each of the forward rates is plotted in blue and is near 100% for all 179 quarterly segments of the yield curve. The one factor model in red, by contrast, does a poor job of fitting 91-day movements in the quarterly forward rates. The adjusted r-squared is good, of course, for the first forward rate since the short rate is the standard risk factor in a one factor term structure model. Beyond the first quarter, however, explanatory power declines rapidly. The adjusted r-squared of the one factor model never exceeds 5% after the first 15 quarterly forward rates and is below that level at most maturities.

The root mean squared error for the 1, 2, 3, 6 and 11 factor constant coefficient model is shown in Exhibit VIII.

Exhibit VIII



The root mean squared error for the 11 factor model is less than 0.01% at every maturity along the yield curve. This result should not come as a surprise to a serious analyst, because it is very similar to the results of the best practice Heath, Jarrow and Morton term structure models for U.S. Treasuries, Government of Canada Bonds, United Kingdom Government Bonds, German Bunds, Australian Commonwealth Government Securities, Singapore Government Securities, Spanish Government Securities, Swedish Government Securities, and Japanese Government Bond yields.

IV. Conclusion

The Thai Government Bond yield curve is driven by 11 factors, a number of factors very similar to government yield curves in nine other markets for which studies have been conducted. The 1999-2016 yield history for Thailand is both relatively short and spans a fairly narrow range of the interest rate variation seen internationally. This narrow range makes it more likely that rate-sensitive stochastic volatility is hard to detect. That is indeed the case for Thailand.

The task of estimating interest rate volatility was slightly complicated by the high degree of granularity of the yields reported by the Thai Bond Market Association. Standard model validation procedures revealed that this highly granular yield data implied implausible variation in forward rates, which would distort measured interest rate volatility. To avoid this, we restricted the long term yields used as input to the maximum smoothness forward rate process to maturities close to the maturities at which the Thai Government is a regular bond issuer. Given this modified data set, the constant volatility assumption provided more accurate and more reasonable parameters than a stochastic volatility model. To date, the Thai Government Bond market is the only market of the 10 studied so far for which this is true. We speculate that a stochastic volatility model will ultimately be more accurate as the data series lengthens and the range of Thai interest rate experience widens.

Appendix

In spite of the overwhelming evidence across countries that government bond yields are driven by multiple factors, the use of single factor term structure models in interest rate risk management systems remains common even in some of the world's largest banks. This appendix asks and answers a number of important issues in the use of one factor models that any sophisticated model audit would pose. Given the answers below, most analysts would conclude that one factor term structure models are less accurate than a long list of multi-factor term structure models and that the one factor models would therefore fail a model audit.

We address two classes of one factor term structure models, all of which are special cases of the Heath, Jarrow and Morton framework, in this appendix using data from the Thai Government Bond market. Answers for other government bond markets cited in the references are nearly identical.

One factor models with rate-dependent interest rate volatility;

- Cox, Ingersoll and Ross (1985)
- Black, Derman and Toy (1990)
- Black and Karasinski (1991)

One factor models with constant interest rate volatility (affine models)

- Vasicek (1977)
- Ho and Lee (1986)
- Extended Vasicek or Hull and White Model (1990, 1993)

Non-parametric test 1: Can interest rates be negative in the model?

The one factor models with rate-dependent interest rate volatility make it impossible for interest rates to be negative. Is this implication true or false? It is false, as Deutsche Bundesbank yield histories, Swedish Government Bond histories, Japanese Government Bond histories, and yields in many other countries show frequent negative yields in in recent years. As yet, negative rates have not been experienced in the Thai Government Bond market. This video of forward rates and zero coupon bond for the Japanese Government Bond yield curve documents the existence of negative forward rates using daily data from September 24, 1974 through December 30, 2016:

https://www.youtube.com/watch?v=X49I1rIZPJg

Non-parametric test 2: As commonly implemented, one factor term structure models imply that all yields will either (a) rise, (b) fall, or (c) remain unchanged. This implication is false, as documented for Thailand in Table III. In fact, yield curves have twisted on 93% of the observations for the Thai Government Bond market.

Non-parametric test 3: The constant coefficient one-factor models imply that zero coupon yields are normally distributed and so are the changes in zero coupon yields. In the Thai Government Bond market, this implication is rejected by three common statistical tests for 172 of 180 quarterly maturities for zero yields and for all 179 of the quarterly changes, as shown in Table II.

Assertion A: There are no factors other than the short term rate of interest that are statistically significant in explaining yield curve movements. This assertion is false. Table V shows, using principal components analysis, that 12 factors are needed to explain the movements of the Thai Government Bond yield curve.

Assertion B: There may be more than one factor, but the incremental explanatory power of the 2nd and other factors is so miniscule as to be useless. This assertion is false, as the 2nd through 12th factors in the Thai Government Bond market explain 64% of forward rate movements, compared to 36% for the first factor alone.

Assertion C: A one-factor "regime shift" model is all that is necessary to match the explanatory power of the 2nd and other factors. This assertion is also false. A recent study prepared for a major U.S. bank regulator also confirmed that a one factor "regime shift" term structure model made essentially no incremental contribution toward resolving the persistent lack of accuracy in one factor term structure models.
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